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Wind Effects on the Water in a Narrow Two-Layered Lake

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Phil. Trans. R. Soc. Lond. A 1966 **259**, 391-430
doi: 10.1098/rsta.1966.0021

WIND EFFECTS ON THE WATER IN A NARROW TWO-LAYERED LAKE

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(*Communicated by J. Proudman, F.R.S.—Received* 25 *March* 1965)

CONTENTS

This paper presents a theoretical study of water movement in a long narrow lake subject to wind action during the summer season of thermal stratification. A model basin of uniform depth and width, consisting of two homogeneous layers of slightly different density, is considered. The motion of the water is assumed to be two dimensional in the vertical longitudinal section; geostrophic effects are ignored. The top and bottom layers in the model respectively represent the relatively warm surface water and the colder bottom water in the natural lake.

Hydrodynamical equations are formulated in terms of the currents in the upper and lower layers, the elevation of the interface between the layers, and the elevation of the water surface. Solutions are sought to determine the dynamic response of the basin to an instantaneous rise in the wind stress applied tangentially over the surface. Three cases are considered corresponding to different frictional conditions at the bottom of the basin: (i) bottom friction zero, (ii) bottom friction proportional to the depth mean of the horizontal current in the lower layer, (iii) bottom current zero. It is assumed that internal friction is zero at the interface between the layers (this interface corresponds to the thermocline boundary in reality).

Results obtained show that in the motion of the water there are ordinary and internal seiches characteristic of the two-layered model, together with a wind-driven circulation in the top layer. The theory is applied to determine vertical oscillations of the thermocline in an actual lake (Windermere) at one station, in response to a succession of wind pulses representing actual wind conditions over the lake. The oscillations thus obtained from theory compare satisfactorily with those derived from temperature observations taken in the lake. Depth-mean currents in the lake are deduced from theory, but there are no current measurements against which these values may be tested.

The paper is divided into three parts. Part I deals with the development of the theory. Part II gives an account of actual physical conditions in Windermere, describing the analysis of temperature observations taken in the lake (yielding thermocline movements) and the analysis of wind records (yielding corresponding values of wind stress over the water surface). Part III is concerned with the numerical application of the theory to Windermere (under conditions described in part II), and gives general conclusions resulting from the entire work.

PART I. THEORETICAL ANALYSIS

By N. S. Heaps

1. Introduction

Probably the most striking feature of movement in a narrow stratified lake in its response to the wind is the oscillatory motion of the isotherms at the level of the thermocline (Mortimer 1953). On the basis of a simple two-layered theoretical model of the lake, the oscillations may be explained in terms of internal standing waves, or seiches (Proudman 1953, pp. 336–341). Previous work has been mainly concerned with the determination of the seiche periods and modes, and with motion consisting of these modes when atmospheric disturbances are absent (Mortimer 1952). In the present paper the dynamic response of a two-layered lake to wind stress is investigated theoretically; Proudman & Doodson (1924) have already given a theoretical account of meteorological effects in a sea of *uniform* density. The amplitudes of the ordinary and internal seiches set up in the lake are related to the wind stress, and the characteristics of wind-driven circulations in the surface layer are examined. The theory is used to deduce the fluctuations in level of the thermocline in Windermere in response to a series of wind-stress pulses; the results thus obtained compare satisfactorily with fluctuations observed in nature corresponding to approximately the same wind-stress variation.

The following assumptions are made in the analysis:

(i) The lake, in equilibrium, consists of two horizontal layers of water, each homogeneous, with different densities. The equilibrium depths of the layers do not vary with time.

(ii) The lake is long and narrow. Its longitudinal section is rectangular and the depth is small in comparison with the length.

(iii) There is no motion of the water in a transverse direction: the motion is two-dimensional in a longitudinal section of the lake.

(iv) Velocity components and vertical displacements are small enough for terms involving their squares and products to be ignored.

(v) Vertical acceleration is negligible.

(vi) Geostrophic effects may be ignored.

(vii) Of the components of internal stress at any point in the water, only the shear stress acting over a horizontal plane in the longitudinal direction is of importance in the equations of motion.

(viii) Eddy viscosity is constant and uniform within each layer, but, in general, differs as between layers.

(ix) Internal friction at the interface between the layers is zero.

(x) External friction is applied to the water at the bottom of the lake; friction at the sides of the lake is ignored.

(xi) Wind stress over the water surface varies with the square of the wind speed and with a drag coefficient that increases linearly with the wind speed for speeds between 5 and 20 m/s (Francis 1959).

(xii) Atmospheric pressure over the water surface is constant and uniform.

(xiii) The water is incompressible.

In the thermocline region of a stratified lake the high vertical density gradient tends to suppress vertical turbulence and the associated internal friction (Mortimer 1953, p. 99). Assumption (ix) represents this condition in the two-layered model. The assumption has been widely used in oceanography (see for example, Defant 1961, p. 405; Proudman 1953, p. 102). Present knowledge concerning turbulence at the various levels in a stratified lake subject to wind stress is, however, imperfect (Mortimer 1961): future advances in this field may therefore lead, in due course, to a modification of (ix).

2. Equations for a two-layered lake

The lake is of length l and rectangular axes Ox, Oz are taken within a vertical longitudinal section (figure 1). The origin O is a point at one end of the lake in the undisturbed water surface: Ox, horizontal and directed along the length, lies within this surface; Oz acts vertically downwards.

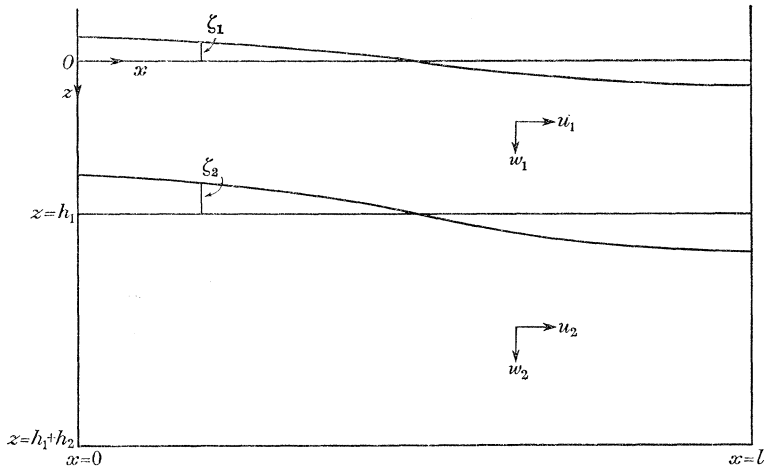

Figure 1. A longitudinal section of the two-layered lake with the water in motion, showing velocity components and elevations of the upper surface and the surface of discontinuity.

Under the assumptions listed in §1 the hydrodynamical equations governing the motion of water of uniform constant density ρ in the lake (Proudman 1953, pp. 88, 96) reduce to

$$\frac{\partial u}{\partial x}+\frac{\partial w}{\partial z}=0, \tag{1}$$

$$\frac{\partial u}{\partial t}=-\frac{1}{\rho}\left(\frac{\partial p}{\partial x}+\frac{\partial F_{zx}}{\partial z}\right), \tag{2}$$

$$0=-\frac{1}{\rho}\frac{\partial p}{\partial z}+g, \tag{3}$$

where u, w, are, respectively, the components of current in the directions of x, z increasing; p is the pressure at a point in the water; F_{zx} the frictional stress, in the x direction, exerted by the water above a depth z on the water below that depth; g the acceleration due to the Earth's gravity; and t the time.

When the water is at rest a homogeneous layer of density ρ_1 lies to a depth h_1 over another homogeneous layer of density ρ_2 and depth h_2, the total depth of the lake being $h\,(=h_1+h_2)$. When there is motion the elevation of the upper surface is ζ_1, and that of the surface of discontinuity ζ_2 (figure 1). If we take

$$u=u_1,\quad w=w_1,\quad p=p_1,\quad F_{zx}=F_1,\quad \text{in the upper layer}\quad (-\zeta_1\leqslant z\leqslant h_1-\zeta_2),$$

$$u=u_2,\quad w=w_2,\quad p=p_2,\quad F_{zx}=F_2,\quad \text{in the lower layer}\quad (h_1-\zeta_2\leqslant z\leqslant h),$$

it follows from (1), (2) and (3) that, for the upper layer,

$$\frac{\partial u_1}{\partial x}+\frac{\partial w_1}{\partial z}=0, \tag{4}$$

$$\frac{\partial u_1}{\partial t}=-\frac{1}{\rho_1}\left(\frac{\partial p_1}{\partial x}+\frac{\partial F_1}{\partial z}\right), \tag{5}$$

$$\frac{\partial p_1}{\partial z}=\rho_1 g, \tag{6}$$

and for the lower layer,

$$\frac{\partial u_2}{\partial x}+\frac{\partial w_2}{\partial z}=0, \tag{7}$$

$$\frac{\partial u_2}{\partial t}=-\frac{1}{\rho_2}\left(\frac{\partial p_2}{\partial x}+\frac{\partial F_2}{\partial z}\right), \tag{8}$$

$$\frac{\partial p_2}{\partial z}=\rho_2 g. \tag{9}$$

If p_a denotes atmospheric pressure, assumed constant and uniform, the condition on the pressure at the surface of the water is

$$p_1=p_a \quad \text{when} \quad z=-\zeta_1.$$

For pressure to be continuous at the surface of discontinuity,

$$p_1=p_2 \quad \text{when} \quad z=h_1-\zeta_2.$$

Integrating (6) and (9), satisfying these conditions, gives

$$\left.\begin{aligned} p_1&=p_a+g\rho_1(\zeta_1+z),\\ p_2&=p_a+g\rho_1(\zeta_1+h_1-\zeta_2)+g\rho_2(\zeta_2+z-h_1).\end{aligned}\right\} \tag{10}$$

Substituting (10) into (5) and (8) we get

$$\frac{\partial u_1}{\partial t}=-g\frac{\partial \zeta_1}{\partial x}-\frac{1}{\rho_1}\frac{\partial F_1}{\partial z}, \tag{11}$$

$$\frac{\partial u_2}{\partial t}=-g\frac{\rho_1}{\rho_2}\frac{\partial \zeta_1}{\partial x}-g\left(1-\frac{\rho_1}{\rho_2}\right)\frac{\partial \zeta_2}{\partial x}-\frac{1}{\rho_2}\frac{\partial F_2}{\partial z}. \tag{12}$$

It is assumed that ζ_1, ζ_2 (functions of x and t) are small enough for u, w, F_{zx} at the upper surface $z=-\zeta_1$ to be evaluated on $z=0$, and for u, w, F_{zx} at the surface of discontinuity $z=h_1-\zeta_2$ to be evaluated on $z=h_1$, throughout the motion. This means that in the mathematical analysis u_1, w_1, F_1 are defined for $0\leqslant z\leqslant h_1$, and u_2, w_2, F_2 for $h_1\leqslant z\leqslant h$, throughout the motion. Then, integrating (4) and (11) with respect to z between $z = 0$ and $z = h_1$, and (7) and (12) with respect to z between $z = h_1$ and $z = h$, noting that

$$\left.\begin{aligned} w_1(z=0) &= -\partial\zeta_1/\partial t, \\ w_1(z=h_1) = w_2(z=h_1) &= -\partial\zeta_2/\partial t, \\ w_2(z=h) &= 0, \end{aligned}\right\} \tag{13}$$

we have
$$\frac{\partial}{\partial x}\int_0^{h_1} u_1\,\mathrm{d}z + \frac{\partial\zeta_1}{\partial t} - \frac{\partial\zeta_2}{\partial t} = 0, \tag{14}$$

$$\frac{\partial}{\partial t}\int_0^{h_1} u_1\,\mathrm{d}z = -gh_1\frac{\partial\zeta_1}{\partial x} + \frac{1}{\rho_1}(F_S - F_D), \tag{15}$$

$$\frac{\partial}{\partial x}\int_{h_1}^{h} u_2\,\mathrm{d}z + \frac{\partial\zeta_2}{\partial t} = 0, \tag{16}$$

$$\frac{\partial}{\partial t}\int_{h_1}^{h} u_2\,\mathrm{d}z = -gh_2\frac{\rho_1}{\rho_2}\frac{\partial\zeta_1}{\partial x} - gh_2\left(1 - \frac{\rho_1}{\rho_2}\right)\frac{\partial\zeta_2}{\partial x} + \frac{1}{\rho_2}(F_D - F_B), \tag{17}$$

where

$F_S = F_{zx}(z=0)$ = component of wind stress over the surface of the lake in the direction Ox,

$F_D = F_{zx}(z=h_1)$ = internal friction at the surface of discontinuity,

$F_B = F_{zx}(z=h)$ = bottom friction.

The depth-mean values of u_1, u_2 are

$$u_{1m} = \frac{1}{h_1}\int_0^{h_1} u_1\,\mathrm{d}z, \quad u_{2m} = \frac{1}{h_2}\int_{h_1}^{h} u_2\,\mathrm{d}z, \tag{18}$$

and therefore, (14) to (17) may be written

$$h_1\frac{\partial u_{1m}}{\partial x} + \frac{\partial\zeta_1}{\partial t} - \frac{\partial\zeta_2}{\partial t} = 0, \tag{19}$$

$$\frac{\partial u_{1m}}{\partial t} = -g\frac{\partial\zeta_1}{\partial x} + \frac{1}{\rho_1 h_1}(F_S - F_D), \tag{20}$$

$$h_2\frac{\partial u_{2m}}{\partial x} + \frac{\partial\zeta_2}{\partial t} = 0, \tag{21}$$

$$\frac{\partial u_{2m}}{\partial t} = -g\frac{\rho_1}{\rho_2}\frac{\partial\zeta_1}{\partial x} - g\left(1 - \frac{\rho_1}{\rho_2}\right)\frac{\partial\zeta_2}{\partial x} + \frac{1}{\rho_2 h_2}(F_D - F_B). \tag{22}$$

When there is no wind stress and no internal friction, then $F_S = F_D = F_B = 0$ and (19) to (22) reduce to the equations governing surface and internal seiches given by Proudman (1953, p. 338). In this latter case, u_1 and u_2 are independent of depth z so that $u_1 = u_{1m}$, $u_2 = u_{2m}$. From (11), (12), (14) and (16), writing

$$F_1 = -\rho_1 N_1\frac{\partial u_1}{\partial z}, \quad F_2 = -\rho_2 N_2\frac{\partial u_2}{\partial z}, \tag{23}$$

where N_1, N_2 are coefficients of eddy viscosity (assumed constant and uniform) in the upper and lower layers respectively, we get

$$\frac{\partial u_1}{\partial t} = -g\frac{\partial \zeta_1}{\partial x} + N_1\frac{\partial^2 u_1}{\partial z^2}, \tag{24}$$

$$\frac{\partial u_2}{\partial t} = -g\frac{\rho_1}{\rho_2}\frac{\partial \zeta_1}{\partial x} - g\left(1-\frac{\rho_1}{\rho_2}\right)\frac{\partial \zeta_2}{\partial x} + N_2\frac{\partial^2 u_2}{\partial z^2}, \tag{25}$$

$$\frac{\partial}{\partial x}\int_0^{h_1} u_1 \,\mathrm{d}z + \frac{\partial \zeta_1}{\partial t} - \frac{\partial \zeta_2}{\partial t} = 0, \tag{26}$$

$$\frac{\partial}{\partial x}\int_{h_1}^{h} u_2 \,\mathrm{d}z + \frac{\partial \zeta_2}{\partial t} = 0. \tag{27}$$

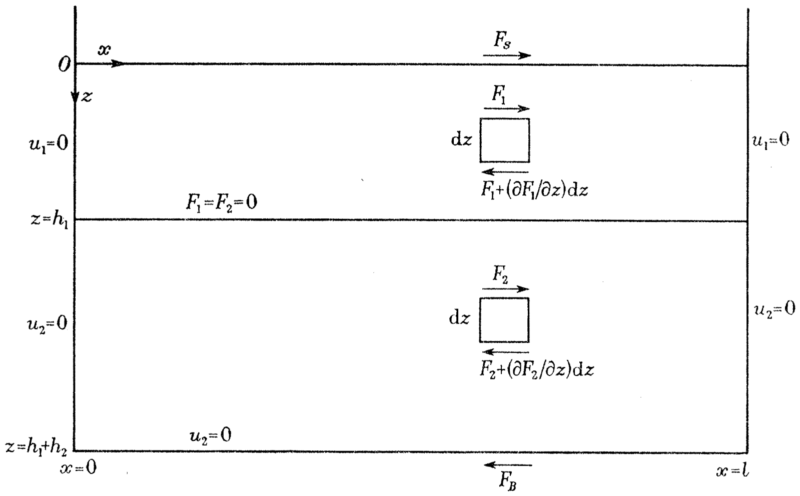

FIGURE 2. A longitudinal section of the two-layered lake showing horizontal shear stress and boundary conditions.

To determine the motion in the lake, (24) to (27) have to be solved for $u_1, u_2, \zeta_1, \zeta_2$; boundary conditions may be taken as follows (figure 2):

$$u_1 = u_2 = 0 \quad \text{at} \quad x = 0, \quad x = l, \tag{28}$$

$$F_1 = F_S, \quad \text{i.e.} \quad -\rho_1 N_1\, \partial u_1/\partial z = F_S, \quad \text{at} \quad z = 0, \tag{29}$$

$$F_1 = F_2 = 0, \quad \text{i.e.} \quad \partial u_1/\partial z = \partial u_2/\partial z = 0, \quad \text{at} \quad z = h_1, \tag{30}$$

$$u_2 = 0 \quad \text{at} \quad z = h. \tag{31}$$

Condition (30) states that internal friction at the surface of discontinuity is zero, i.e. the water of the top layer may slide freely over the water of the bottom layer at this level. Condition (31), postulating zero bottom current, has been used by Proudman & Doodson (1924); it is employed for mathematical convenience instead of the more satisfactory frictional condition

$$-\rho_2 N_2\, \partial u_2/\partial z = K\rho_2 u_2|u_2| \quad \text{at} \quad z = h,$$

where K is a constant taken to be 0·002 by Jeffreys (1923), and 0·0025 by Proudman (1953, p. 136). On the assumption that the bottom friction is zero, (31) may be replaced by

$$\partial u_2/\partial z = 0 \quad \text{at} \quad z = h. \tag{32}$$

An approximate form for the bottom friction, suitable for use in the solution of equations (19) to (22), is given by

$$F_B/\rho_2 h_2 = 2ku_{2m}, \tag{33}$$

where k is some constant.

3. Summary of procedure

In the mathematics that follows, solutions of the hydrodynamical equations formulated in § 2 are sought, to determine the motion produced in the two-layered lake by an instantaneous rise in the wind stress F_S from zero to a stationary distribution expressed as a Fourier series in x. Three cases are considered corresponding to different frictional conditions at the bottom of the lake; the internal friction at the surface of discontinuity between the layers is assumed to be zero.

First, in § 4, equations (24) to (27) are solved with zero bottom friction; the boundary conditions employed are given by (28), (29), (30) and (32). Expressions are obtained for the elevations ζ_1, ζ_2 and the horizontal currents u_1, u_2 which indicate the main features of the motion in the lake.

Secondly, in § 5, bottom friction is taken into account. To simplify the work, equations (19) to (22) are solved with $F_D = 0$ and F_B given by (33). These equations do not involve the z coordinate and in their solution approximate expressions are obtained for the elevations ζ_1, ζ_2 and the depth-mean currents u_{1m}, u_{2m}.

Thirdly, in § 6, equations (24) to (27) are again solved, this time assuming that the bottom current is zero. The boundary conditions are given by (28) to (31). Thus, the elevations ζ_1, ζ_2 and the currents at any depth u_1, u_2 may be determined, subject to the influence of bottom friction. Although this solution may be regarded as the most general obtained in the three cases considered, the mathematical analysis is complicated. The results are interpreted in the light of the knowledge gained from the simpler analyses of §§ 4 and 5.

The response of the lake to an instantaneous rise in the wind stress F_S having been determined, its response when F_S varies continuously with time may be deduced by integration, as outlined in § 7.

In part III, the theories developed in §§ 5 and 6 are applied to Windermere during a period of summer stratification. The thermocline in the lake corresponds to the density discontinuity in the two-layered theoretical model: fluctuations in the level of the thermocline are, therefore, represented theoretically by the variation of ζ_2.

4. Solution of the equations with zero bottom friction

(a) *A sine-wave distribution of wind stress*

The response of the two-layered lake described in § 2 to a sudden increase in the wind stress F_S at time $t = 0$ is now determined. We take

$$F_S = H(t)\, A_n \sin(n\pi x/l), \tag{34}$$

where $H(t)$ is Heaviside's unit function (Carslaw & Jaeger 1947, p. 7), n a positive integer, and A_n a constant. According to (34), the wind stress F_S rises instantaneously at $t=0$ from zero to $A_n \sin n\pi x/l$ $(0 \leqslant x \leqslant l)$ and subsequently remains constant. A solution of (24) to (27) is sought of the form

$$\left.\begin{aligned} \zeta_1 &= Z_1(t)\cos(n\pi x/l),\\ \zeta_2 &= Z_2(t)\cos((n\pi x/l),\\ u_1 &= U_1(z,t)\sin(n\pi x/l),\\ u_2 &= U_2(z,t)\sin(n\pi x/l).\end{aligned}\right\} \tag{35}$$

The boundary condition (28) is then satisfied, and (35) in (24) to (27) gives

$$\left.\begin{aligned} \frac{\partial U_1}{\partial t} &= \frac{n\pi g}{l}Z_1 + N_1\frac{\partial^2 U_1}{\partial z^2},\\ \frac{\partial U_2}{\partial t} &= \frac{n\pi g}{l}\frac{\rho_1}{\rho_2}Z_1 + \frac{n\pi g}{l}\left(1-\frac{\rho_1}{\rho_2}\right)Z_2 + N_2\frac{\partial^2 U_2}{\partial z^2},\\ \frac{n\pi}{l}\int_0^{h_1} U_1\,\mathrm{d}z &+ \frac{\mathrm{d}Z_1}{\mathrm{d}t} - \frac{\mathrm{d}Z_2}{\mathrm{d}t} = 0,\\ \frac{n\pi}{l}\int_{h_1}^{h} U_2\,\mathrm{d}z &+ \frac{\mathrm{d}Z_2}{\mathrm{d}t} = 0.\end{aligned}\right\} \tag{36}$$

If initially the water in the lake lies at rest under no wind forces, so that

$$U_1 = U_2 = Z_1 = Z_2 = 0 \quad \text{when} \quad t=0,$$

taking Laplace transforms in (36) (Churchill 1958, p. 3) yields

$$s\bar{U}_1 = \frac{n\pi g}{l}\bar{Z}_1 + N_1\frac{\mathrm{d}^2\bar{U}_1}{\mathrm{d}z^2}, \tag{37}$$

$$s\bar{U}_2 = \frac{n\pi g}{l}\frac{\rho_1}{\rho_2}\bar{Z}_1 + \frac{n\pi g}{l}\left(1-\frac{\rho_1}{\rho_2}\right)\bar{Z}_2 + N_2\frac{\mathrm{d}^2\bar{U}_2}{\mathrm{d}z^2}, \tag{38}$$

$$\frac{n\pi}{l}\int_0^{h_1}\bar{U}_1\,\mathrm{d}z + s\bar{Z}_1 - s\bar{Z}_2 = 0, \tag{39}$$

$$\frac{n\pi}{l}\int_{h_1}^{h}\bar{U}_2\,\mathrm{d}z + s\bar{Z}_2 = 0, \tag{40}$$

where the Laplace transform of a function $Q(z,t)$ is defined by

$$\bar{Q}(z,s) = \int_0^\infty \mathrm{e}^{-st}Q(z,t)\,\mathrm{d}t.$$

Solutions of (37) and (38) are:

$$\left.\begin{aligned} \bar{U}_1 &= B_1\cosh\alpha_1 z + C_1\sinh\alpha_1 z + \frac{n\pi g}{ls}\bar{Z}_1,\\ \bar{U}_2 &= B_2\cosh\alpha_2 z + C_2\sinh\alpha_2 z + \frac{n\pi g}{ls}\frac{\rho_1}{\rho_2}\bar{Z}_1 + \frac{n\pi g}{ls}\left(1-\frac{\rho_1}{\rho_2}\right)\bar{Z}_2,\end{aligned}\right\} \tag{41}$$

where B_1, C_1, B_2, C_2 are arbitrary constants, and

$$\alpha_1 = (s/N_1)^{\frac{1}{2}}, \quad \alpha_2 = (s/N_2)^{\frac{1}{2}}. \tag{42}$$

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It is assumed that the bottom friction is zero, also the internal friction at the surface of discontinuity between the layers of the lake; the boundary conditions given by (29), (30) and (32) are, therefore, applicable. Taking Laplace transforms in (29), (30) and (32), using (34) and (35), we get

$$\left.\begin{aligned} -\rho_1 N_1 \frac{\mathrm{d}\bar{U}_1}{\mathrm{d}z} &= \frac{A_n}{s} \quad \text{at} \quad z=0,\\ \frac{\mathrm{d}\bar{U}_1}{\mathrm{d}z} = \frac{\mathrm{d}\bar{U}_2}{\mathrm{d}z} &= 0 \quad \text{at} \quad z=h_1,\\ \frac{\mathrm{d}\bar{U}_2}{\mathrm{d}z} &= 0 \quad \text{at} \quad z=h. \end{aligned}\right\} \tag{43}$$

Then, substituting $\bar{U}_1$, $\bar{U}_2$ given by (41) into (39), (40) and (43), gives

$$\left(\frac{n^2\pi^2 g h_1}{l^2}+s^2\right)\bar{Z}_1 - s^2\bar{Z}_2 = \frac{n\pi s}{l\alpha_1}[-B_1\sinh\alpha_1 h_1 + C_1(1-\cosh\alpha_1 h_1)], \tag{44}$$

$$\frac{n^2\pi^2 g h_2}{l^2}\frac{\rho_1}{\rho_2}\bar{Z}_1 + \left[\frac{n^2\pi^2 g h_2}{l^2}\left(1-\frac{\rho_1}{\rho_2}\right)+s^2\right]\bar{Z}_2$$

$$= \frac{n\pi s}{l\alpha_2}[B_2(\sinh\alpha_2 h_1 - \sinh\alpha_2 h) + C_2(\cosh\alpha_2 h_1 - \cosh\alpha_2 h)], \tag{45}$$

$$-\rho_1 N_1 \alpha_1 C_1 = A_n/s, \tag{46}$$

$$B_1\sinh\alpha_1 h_1 + C_1\cosh\alpha_1 h_1 = 0, \tag{47}$$

$$B_2\sinh\alpha_2 h_1 + C_2\cosh\alpha_2 h_1 = 0, \tag{48}$$

$$B_2\sinh\alpha_2 h + C_2\cosh\alpha_2 h = 0. \tag{49}$$

It follows immediately from (46) to (49) that

$$B_1 = \frac{A_n\cosh\alpha_1 h_1}{s\rho_1 N_1\alpha_1\sinh\alpha_1 h_1}, \quad C_1 = -\frac{A_n}{s\rho_1 N_1\alpha_1}, \quad B_2 = C_2 = 0.$$

Then, from (44), (45) and (41),

$$\left.\begin{aligned} \bar{Z}_1 &= -\frac{n\pi A_n}{l\rho_1}\left[\frac{n^2\pi^2 g h_2}{l^2}\left(1-\frac{\rho_1}{\rho_2}\right)+s^2\right]\Big/ s\Phi(s),\\ \bar{Z}_2 &= \left(\frac{n^3\pi^3 g h_2 A_n}{l^3\rho_2}\right)\Big/ s\Phi(s),\\ \bar{U}_1 &= -\frac{n^2\pi^2 g A_n}{l^2\rho_1}\left[\frac{n^2\pi^2 g h_2}{l^2}\left(1-\frac{\rho_1}{\rho_2}\right)+s^2\right]\Big/ s^2\Phi(s) + \frac{A_n\cosh\{\alpha_1(h_1-z)\}}{s\rho_1 N_1\alpha_1\sinh\alpha_1 h_1},\\ \bar{U}_2 &= -\left(\frac{n^2\pi^2 g A_n}{l^2\rho_2}\right)\Big/\Phi(s). \end{aligned}\right\} \tag{50}$$

Here,

$$\Phi(s) = s^4 + \frac{n^2\pi^2 g(h_1+h_2)}{l^2}s^2 + \frac{n^4\pi^4 g^2 h_1 h_2}{l^4}\left(1-\frac{\rho_1}{\rho_2}\right)$$

$$= \left(s^2+\frac{n^2\pi^2\beta_1^2}{l^2}\right)\left(s^2+\frac{n^2\pi^2\beta_2^2}{l^2}\right), \tag{51}$$

where

$$\left.\begin{matrix}\beta_1^2\\ \beta_2^2\end{matrix}\right\} = \tfrac{1}{2}g[h_1+h_2 \pm \{(h_1+h_2)^2 - 4h_1h_2(1-\rho_1/\rho_2)\}^{\frac{1}{2}}]. \tag{52}$$

Clearly, from (52),

$$\beta_i^4 - g(h_1+h_2)\,\beta_i^2 + g^2 h_1 h_2 (1-\rho_1/\rho_2) = 0 \quad (i = 1, 2). \tag{53}$$

Relations (51) and (53) enable us to express the transforms $\bar{Z}_1, \bar{Z}_2, \bar{U}_1, \bar{U}_2$, given by (50), as follows:

$$\bar{Z}_1 = -\left(\frac{lA_n}{n\pi\rho_1 h_1 g}\right)\frac{1}{s} + \frac{lA_n}{n\pi\rho_1 h_1 g(\beta_1^2-\beta_2^2)}\left[(\beta_1^2-gh_2)\frac{s}{s^2+n^2\pi^2\beta_1^2/l^2} - (\beta_2^2-gh_2)\frac{s}{s^2+n^2\pi^2\beta_2^2/l^2}\right],$$

$$\bar{Z}_2 = \left(\frac{lA_n}{n\pi(\rho_2-\rho_1)\,h_1 g}\right)\frac{1}{s} + \frac{h_2 glA_n}{n\pi\rho_2(\beta_1^2-\beta_2^2)}\left[\frac{1}{\beta_1^2}\frac{s}{s^2+n^2\pi^2\beta_1^2/l^2} - \frac{1}{\beta_2^2}\frac{s}{s^2+n^2\pi^2\beta_2^2/l^2}\right],$$

$$\bar{U}_1 = -\left(\frac{A_n}{\rho_1 h_1}\right)\frac{1}{s^2} + \frac{A_n \cosh\{\alpha_1(h_1-z)\}}{s\rho_1 N_1 \alpha_1 \sinh \alpha_1 h_1}$$
$$+\frac{lA_n}{n\pi\rho_1 h_1(\beta_1^2-\beta_2^2)}\left[\frac{1}{\beta_1}(\beta_1^2-gh_2)\frac{n\pi\beta_1/l}{s^2+n^2\pi^2\beta_1^2/l^2} - \frac{1}{\beta_2}(\beta_2^2-gh_2)\frac{n\pi\beta_2/l}{s^2+n^2\pi^2\beta_2^2/l^2}\right],$$

$$\bar{U}_2 = \frac{glA_n}{n\pi\rho_2(\beta_1^2-\beta_2^2)}\left[\frac{1}{\beta_1}\frac{n\pi\beta_1/l}{s^2+n^2\pi^2\beta_1^2/l^2} - \frac{1}{\beta_2}\frac{n\pi\beta_2/l}{s^2+n^2\pi^2\beta_2^2/l^2}\right], \tag{54}$$

from which, using tables of Laplace transforms and the inversion theorem for the Laplace transformation (Churchill 1958, pp. 17, 176) we deduce

$$Z_1 = -\frac{lA_n}{n\pi\rho_1 h_1 g} + \frac{lA_n}{n\pi\rho_1 h_1 g(\beta_1^2-\beta_2^2)}\left[(\beta_1^2-gh_2)\cos\frac{n\pi\beta_1 t}{l} - (\beta_2^2-gh_2)\cos\frac{n\pi\beta_2 t}{l}\right],$$

$$Z_2 = \frac{lA_n}{n\pi(\rho_2-\rho_1)h_1 g} + \frac{h_2 glA_n}{n\pi\rho_2(\beta_1^2-\beta_2^2)}\left[\frac{1}{\beta_1^2}\cos\frac{n\pi\beta_1 t}{l} - \frac{1}{\beta_2^2}\cos\frac{n\pi\beta_2 t}{l}\right],$$

$$U_1 = -\frac{A_n t}{\rho_1 h_1} + I + \frac{lA_n}{n\pi\rho_1 h_1(\beta_1^2-\beta_2^2)}\left[\frac{1}{\beta_1}(\beta_1^2-gh_2)\sin\frac{n\pi\beta_1 t}{l} - \frac{1}{\beta_2}(\beta_2^2-gh_2)\sin\frac{n\pi\beta_2 t}{l}\right],$$

$$U_2 = \frac{glA_n}{n\pi\rho_2(\beta_1^2-\beta_2^2)}\left[\frac{1}{\beta_1}\sin\frac{n\pi\beta_1 t}{l} - \frac{1}{\beta_2}\sin\frac{n\pi\beta_2 t}{l}\right], \tag{55}$$

where
$$I = \frac{A_n}{2\pi i\rho_1 N_1}\int_{\gamma-i\infty}^{\gamma+i\infty}\frac{e^{\lambda t}\cosh\{(h_1-z)\surd(\lambda/N_1)\}}{\lambda(\lambda/N_1)^{\frac{1}{2}}\sinh\{h_1\surd(\lambda/N_1)\}}\,d\lambda \quad (\gamma > 0). \tag{56}$$

To evaluate this integral consider it taken round the closed contour formed by the line $\lambda = \gamma + i\eta(-\surd(R^2-\gamma^2) \leqslant \eta \leqslant \surd(R^2-\gamma^2), R > \gamma)$ and the arc of the circle $|\lambda| = R$ lying to the left of the line. The integrand is a single-valued function of λ with poles at $\lambda = 0$, $-r^2\pi^2 N_1/h_1^2$ $(r = 1, 2, 3 \ldots)$. We take $R = (r+\frac{1}{2})^2\pi^2 N_1/h_1^2$ so that the contour does not pass through any pole. As $r \to \infty$ the integral over the circular arc tends to zero and therefore the integral round the contour tends to I; in the limit, all the poles of the integrand lie within the contour. It follows, from Cauchy's theorem, that I equals $2\pi i$ times the sum of the residues of the integrand at its poles: the contribution to I from the double pole at $\lambda = 0$ is

$$\frac{A_n t}{\rho_1 h_1} + \frac{A_n}{6\rho_1 N_1 h_1}\{3(h_1-z)^2 - h_1^2\},$$

and from the simple pole at $\lambda = -r^2\pi^2 N_1/h_1^2$ is

$$(-1)^{r+1}\frac{2h_1 A_n}{\rho_1 N_1 \pi^2 r^2}\exp\left(-\frac{r^2\pi^2 N_1 t}{h_1^2}\right)\cos\left\{\frac{r\pi}{h_1}(h_1-z)\right\}.$$

Hence

$$I = \frac{A_n t}{\rho_1 h_1} + \frac{A_n}{6\rho_1 N_1 h_1}\{3(h_1-z)^2 - h_1^2\} + \frac{2h_1 A_n}{\rho_1 N_1 \pi^2}\sum_{r=1}^{\infty}\frac{(-1)^{r+1}}{r^2}\exp\left(-\frac{r^2\pi^2 N_1 t}{h_1^2}\right)\cos\left\{\frac{r\pi}{h_1}(h_1-z)\right\}. \tag{57}$$

Then, (57) in (55) gives

$$U_1 = \frac{A_n}{6\rho_1 N_1 h_1}\{3(h_1-z)^2 - h_1^2\} + \frac{2h_1 A_n}{\rho_1 N_1 \pi^2}\sum_{r=1}^{\infty}\frac{(-1)^{r+1}}{r^2}\exp\left(-\frac{r^2\pi^2 N_1 t}{h_1^2}\right)\cos\left\{\frac{r\pi}{h_1}(h_1-z)\right\}$$
$$+\frac{lA_n}{n\pi\rho_1 h_1(\beta_1^2-\beta_2^2)}\left[\frac{1}{\beta_1}(\beta_1^2-gh_2)\sin\frac{n\pi\beta_1 t}{l} - \frac{1}{\beta_2}(\beta_2^2-gh_2)\sin\frac{n\pi\beta_2 t}{l}\right]. \tag{58}$$

From (35), (55) and (58) we have

$$\zeta_1 = \left[-\frac{lA_n}{n\pi\rho_1 h_1 g} + \frac{lA_n}{n\pi\rho_1 h_1 g(\beta_1^2-\beta_2^2)}\left\{(\beta_1^2-gh_2)\cos\frac{n\pi\beta_1 t}{l} - (\beta_2^2-gh_2)\cos\frac{n\pi\beta_2 t}{l}\right\}\right]\cos\frac{n\pi x}{l},$$

$$\zeta_2 = \left[\frac{lA_n}{n\pi(\rho_2-\rho_1)\,h_1 g} + \frac{h_2 g l A_n}{n\pi\rho_2(\beta_1^2-\beta_2^2)}\left\{\frac{1}{\beta_1^2}\cos\frac{n\pi\beta_1 t}{l} - \frac{1}{\beta_2^2}\cos\frac{n\pi\beta_2 t}{l}\right\}\right]\cos\frac{n\pi x}{l},$$

$$u_1 = \left(A_n\sin\frac{n\pi x}{l}\Big/6\rho_1 N_1 h_1\right)\{3(h_1-z)^2-h_1^2\}$$
$$+\left(2h_1 A_n\sin\frac{n\pi x}{l}\Big/\rho_1 N_1\pi^2\right)\sum_{r=1}^{\infty}\frac{(-1)^{r+1}}{r^2}\exp\left(-\frac{r^2\pi^2 N_1 t}{h_1^2}\right)\cos\left\{\frac{r\pi}{h_1}(h_1-z)\right\}$$
$$+\frac{lA_n}{n\pi\rho_1 h_1(\beta_1^2-\beta_2^2)}\left[\frac{1}{\beta_1}(\beta_1^2-gh_2)\sin\frac{n\pi\beta_1 t}{l} - \frac{1}{\beta_2}(\beta_2^2-gh_2)\sin\frac{n\pi\beta_2 t}{l}\right]\sin\frac{n\pi x}{l},$$

$$u_2 = \frac{glA_n}{n\pi\rho_2(\beta_1^2-\beta_2^2)}\left[\frac{1}{\beta_1}\sin\frac{n\pi\beta_1 t}{l} - \frac{1}{\beta_2}\sin\frac{n\pi\beta_2 t}{l}\right]\sin\frac{n\pi x}{l}. \tag{59}$$

These expressions give ζ_1, ζ_2, u_1, u_2 in the motion of the lake produced by the wind stress

$$F_S = H(t)\,A_n\sin(n\pi x/l). \tag{34}$$

(b) *General distribution of wind stress*

We now consider the more general case when

$$F_S = H(t)\,A(x), \tag{60}$$

where $A(x)$ is any function of x $(0 \leqslant x \leqslant l)$ which may be expressed as a Fourier series:

$$A(x) = \sum_{n=1}^{\infty} A_n\sin\frac{n\pi x}{l}\left(A_n = \frac{2}{l}\int_0^l A(x)\sin\frac{n\pi x}{l}\,\mathrm{d}x\right). \tag{61}$$

Substituting (61) in (60) yields

$$F_S = \sum_{n=1}^{\infty} H(t)\,A_n\sin\frac{n\pi x}{l}. \tag{62}$$

Letting $n = 1, 2, 3, \ldots$ in (59) we get $(\zeta_1, \zeta_2, u_1, u_2)$ corresponding to each term of the series

(62). Superimposing these solutions it follows that in the response of the lake to the wind stress (60),

$$\zeta_1 = \sum_{n=1}^{\infty}\left[-\frac{lA_n}{n\pi\rho_1 h_1 g}+\frac{lA_n}{n\pi\rho_1 h_1 g(\beta_1^2-\beta_2^2)}\left\{(\beta_1^2-gh_2)\cos\frac{n\pi\beta_1 t}{l}-(\beta_2^2-gh_2)\cos\frac{n\pi\beta_2 t}{l}\right\}\right]\cos\frac{n\pi x}{l},$$

$$\zeta_2 = \sum_{n=1}^{\infty}\left[\frac{lA_n}{n\pi(\rho_2-\rho_1)h_1 g}+\frac{h_2 g l A_n}{n\pi\rho_2(\beta_1^2-\beta_2^2)}\left\{\frac{1}{\beta_1^2}\cos\frac{n\pi\beta_1 t}{l}-\frac{1}{\beta_2^2}\cos\frac{n\pi\beta_2 t}{l}\right\}\right]\cos\frac{n\pi x}{l},$$

$$u_1 = \frac{A(x)}{6\rho_1 N_1 h_1}\{3(h_1-z)^2-h_1^2\}+\frac{2h_1 A(x)}{\rho_1 N_1\pi^2}\sum_{r=1}^{\infty}\frac{(-1)^{r+1}}{r^2}\exp\left(-\frac{r^2\pi^2 N_1 t}{h_1^2}\right)\cos\left\{\frac{r\pi}{h_1}(h_1-z)\right\}$$
$$+\frac{l}{\pi\rho_1 h_1(\beta_1^2-\beta_2^2)}\sum_{n=1}^{\infty}\frac{A_n}{n}\left[\frac{1}{\beta_1}(\beta_1^2-gh_2)\sin\frac{n\pi\beta_1 t}{l}-\frac{1}{\beta_2}(\beta_2^2-gh_2)\sin\frac{n\pi\beta_2 t}{l}\right]\sin\frac{n\pi x}{l},$$

$$u_2 = \frac{gl}{\pi\rho_2(\beta_1^2-\beta_2^2)}\sum_{n=1}^{\infty}\frac{A_n}{n}\left[\frac{1}{\beta_1}\sin\frac{n\pi\beta_1 t}{l}-\frac{1}{\beta_2}\sin\frac{n\pi\beta_2 t}{l}\right]\sin\frac{n\pi x}{l}. \tag{63}$$

The expressions for ζ_1, ζ_2 thus obtained may also be derived by solving the equations (19) to (22) taking $F_D = F_B = 0$. To find w_1, w_2 we integrate (4) and (7) and obtain

$$\left.\begin{aligned} w_1 &= -\int_h^{h_1}\frac{\partial u_2}{\partial x}\,\mathrm{d}z-\int_{h_1}^{z}\frac{\partial u_1}{\partial x}\,\mathrm{d}z,\\ w_2 &= -\int_h^{z}\frac{\partial u_2}{\partial x}\,\mathrm{d}z.\end{aligned}\right\} \tag{64}$$

Then, substituting for u_1, u_2 from (63) gives

$$w_1 = \frac{A'(x)}{6\rho_1 N_1 h_1}(h_1-z)\{(h_1-z)^2-h_1^2\}+\frac{2h_1^2 A'(x)}{\rho_1 N_1\pi^3}\sum_{r=1}^{\infty}\frac{(-1)^{r+1}}{r^3}\exp\left(-\frac{r^2\pi^2 N_1 t}{h_1^2}\right)\sin\frac{r\pi}{h_1}(h_1-z)$$
$$+\frac{gh_2}{\rho_2(\beta_1^2-\beta_2^2)}\sum_{n=1}^{\infty}A_n\left[\frac{1}{\beta_1}\sin\frac{n\pi\beta_1 t}{l}-\frac{1}{\beta_2}\sin\frac{n\pi\beta_2 t}{l}\right]\cos\frac{n\pi x}{l}$$
$$+\frac{(h_1-z)}{\rho_1 h_1(\beta_1^2-\beta_2^2)}\sum_{n=1}^{\infty}A_n\left[\frac{1}{\beta_1}(\beta_1^2-gh_2)\sin\frac{n\pi\beta_1 t}{l}-\frac{1}{\beta_2}(\beta_2^2-gh_2)\sin\frac{n\pi\beta_2 t}{l}\right]\cos\frac{n\pi x}{l},$$

$$w_2 = \frac{g(h-z)}{\rho_2(\beta_1^2-\beta_2^2)}\sum_{n=1}^{\infty}A_n\left[\frac{1}{\beta_1}\sin\frac{n\pi\beta_1 t}{l}-\frac{1}{\beta_2}\sin\frac{n\pi\beta_2 t}{l}\right]\cos\frac{n\pi x}{l}, \tag{65}$$

where $A'(x) = \mathrm{d}A(x)/\mathrm{d}x$. Also, from (23) and (63),

$$\left.\begin{aligned} F_1 &= \left(\frac{h_1-z}{h_1}\right)A(x)+\frac{2A(x)}{\pi}\sum_{r=1}^{\infty}\frac{(-1)^r}{r}\exp\left(-\frac{r^2\pi^2 N_1 t}{h_1^2}\right)\sin\frac{r\pi}{h_1}(h_1-z),\\ F_2 &= 0.\end{aligned}\right\} \tag{66}$$

We shall now examine the motion in the lake described by (63) and (65). Terms involving

$$\left.\begin{matrix}\sin\\ \cos\end{matrix}\right\}\frac{n\pi\beta_i t}{l}\left.\begin{matrix}\sin\\ \cos\end{matrix}\right\}\frac{n\pi x}{l}$$

indicate the presence of an undamped seiche of period $2l/n\beta_i$ and wavelength $2l/n$; there are an infinite number of different seiches corresponding to $i = 1, 2$; $n = 1, 2, 3, \ldots, \infty$. In a seiche with period $2l/n\beta_1$,

$$\left.\begin{aligned} \zeta_1 &= \frac{lA_n(\beta_1^2-gh_2)}{n\pi\rho_1 h_1 g(\beta_1^2-\beta_2^2)}\cos\frac{n\pi\beta_1 t}{l}\cos\frac{n\pi x}{l},\\ \zeta_2 &= \frac{h_2 glA_n}{n\pi\rho_2\beta_1^2(\beta_1^2-\beta_2^2)}\cos\frac{n\pi\beta_1 t}{l}\cos\frac{n\pi x}{l},\\ u_1 &= \frac{lA_n(\beta_1^2-gh_2)}{n\pi\rho_1 h_1\beta_1(\beta_1^2-\beta_2^2)}\sin\frac{n\pi\beta_1 t}{l}\sin\frac{n\pi x}{l},\\ u_2 &= \frac{glA_n}{n\pi\rho_2\beta_1(\beta_1^2-\beta_2^2)}\sin\frac{n\pi\beta_1 t}{l}\sin\frac{n\pi x}{l}, \end{aligned}\right\} \tag{67}$$

and in a seiche with period $2l/n\beta_2$,

$$\left.\begin{aligned} \zeta_1 &= -\frac{lA_n(\beta_2^2-gh_2)}{n\pi\rho_1 h_1 g(\beta_1^2-\beta_2^2)}\cos\frac{n\pi\beta_2 t}{l}\cos\frac{n\pi x}{l},\\ \zeta_2 &= -\frac{h_2 glA_n}{n\pi\rho_2\beta_2^2(\beta_1^2-\beta_2^2)}\cos\frac{n\pi\beta_2 t}{l}\cos\frac{n\pi x}{l},\\ u_1 &= -\frac{lA_n(\beta_2^2-gh_2)}{n\pi\rho_1 h_1\beta_2(\beta_1^2-\beta_2^2)}\sin\frac{n\pi\beta_2 t}{l}\sin\frac{n\pi x}{l},\\ u_2 &= -\frac{glA_n}{n\pi\rho_2\beta_2(\beta_1^2-\beta_2^2)}\sin\frac{n\pi\beta_2 t}{l}\sin\frac{n\pi x}{l}. \end{aligned}\right\} \tag{68}$$

Now, since $1-\rho_1/\rho_2$ is very small in comparison with unity, from (52),

$$\beta_1^2 \sim g(h_1+h_2), \quad \beta_2^2 \sim \frac{gh_1h_2}{h_1+h_2}\left(1-\frac{\rho_1}{\rho_2}\right). \tag{69}$$

To this order of approximation, it follows that in the mode of oscillation (67),

$$\left.\begin{aligned} T &= \frac{2l}{n\{g(h_1+h_2)\}^{\frac{1}{2}}},\\ \frac{\zeta_2}{\zeta_1} &= \frac{h_2}{h_1+h_2}, \quad \frac{u_2}{u_1} = 1, \end{aligned}\right\} \tag{70}$$

and in the mode of oscillation (68),

$$\left.\begin{aligned} T &= \frac{2l}{n}\left\{\frac{\rho_2}{\rho_2-\rho_1}\left(\frac{1}{h_1}+\frac{1}{h_2}\right)\frac{1}{g}\right\}^{\frac{1}{2}},\\ \frac{\zeta_2}{\zeta_1} &= -\frac{\rho_2}{\rho_2-\rho_1}\left(1+\frac{h_1}{h_2}\right), \quad \frac{u_2}{u_1} = -\frac{h_1}{h_2}, \end{aligned}\right\} \tag{71}$$

where T denotes the period. The relations (70), (71) indicate that the forms for ζ_1, ζ_2, u_1, u_2 given by (67) are those for a surface seiche, and the forms for ζ_1, ζ_2, u_1, u_2 given by (68) are those for an internal seiche (Proudman 1953, p. 340). Hence, in the motion of the lake there are surface and internal seiches: to each positive integer n there is a surface seiche of period $2l/n\beta_1$ and wavelength $2l/n$, and an internal seiche of period $2l/n\beta_2$ also of wavelength $2l/n$.

As shown by (67) and (68) the amplitudes in the various modes of oscillation depend upon the values of A_n which, in turn, depend (according to (61)) on the wind-stress distribution $A(x)$. In particular, if this distribution is symmetrical about $x = \frac{1}{2}l$, then $A_n = 0$ for $n = 2, 4,$

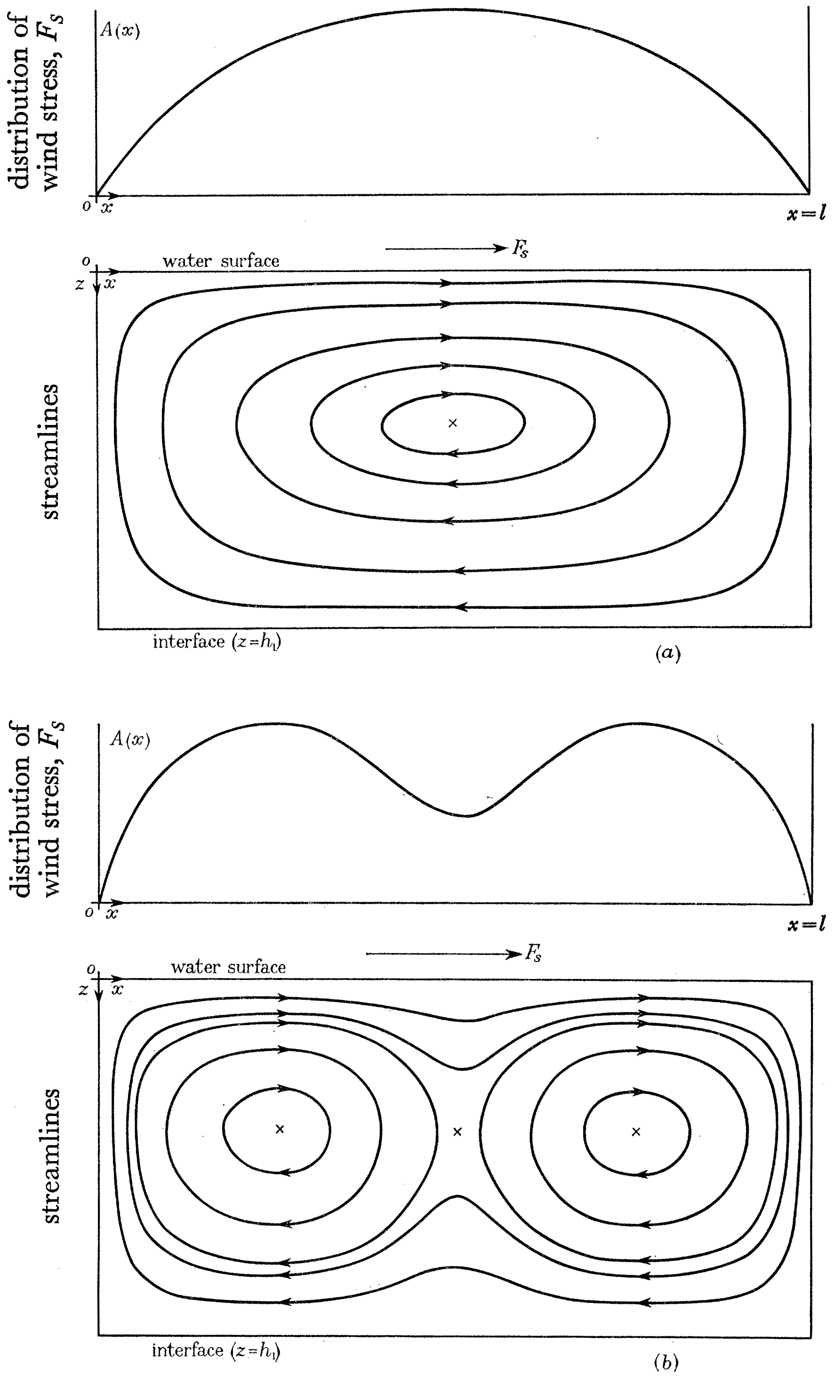

FIGURE 3. Wind-driven circulations in the surface layer: (a) a single circulation, and (b) two interior circulations.

6, etc., i.e. only the seiches corresponding to $n = 1, 3, 5$, etc., are excited; if the distribution is antisymmetrical about $x = \frac{1}{2}l$ then $A_n = 0$ for $n = 1, 3, 5$, etc., and only seiches corresponding to $n = 2, 4, 6$, etc., are present.

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From (63) it follows that the displacements ζ_1, ζ_2 oscillate about steady-state values:

$$\left.\begin{aligned}\zeta_1 &= -\frac{l}{\pi\rho_1 h_1 g}\sum_{n=1}^{\infty}\frac{1}{n}A_n\cos\frac{n\pi x}{l}\quad\left(\frac{\partial\zeta_1}{\partial x}=\frac{A(x)}{\rho_1 h_1 g}\right),\\ \zeta_2 &= \frac{l}{\pi(\rho_2-\rho_1)\,h_1 g}\sum_{n=1}^{\infty}\frac{1}{n}A_n\cos\frac{n\pi x}{l}\quad\left(\frac{\partial\zeta_2}{\partial x}=-\frac{A(x)}{(\rho_2-\rho_1)\,h_1 g}\right).\end{aligned}\right\}\tag{72}$$

As $t\to\infty$, the exponential terms in the expressions for u_1, w_1 ((63), (65)) tend to zero and a steady-state motion is attained in the top layer (in addition to the oscillatory motion) in which

$$\left.\begin{aligned}u_1 &= \frac{A(x)}{6\rho_1 N_1 h_1}\{3(h_1-z)^2-h_1^2\},\\ w_1 &= \frac{A'(x)}{6\rho_1 N_1 h_1}(h_1-z)\{(h_1-z)^2-h_1^2\}.\end{aligned}\right\}\tag{73}$$

Along the streamlines of this motion,

$$\frac{\mathrm{d}z}{\mathrm{d}x}=\frac{w_1}{u_1}=\frac{(h_1-z)\{(h_1-z)^2-h_1^2\}A'(x)}{\{3(h_1-z)^2-h_1^2\}A(x)}.$$

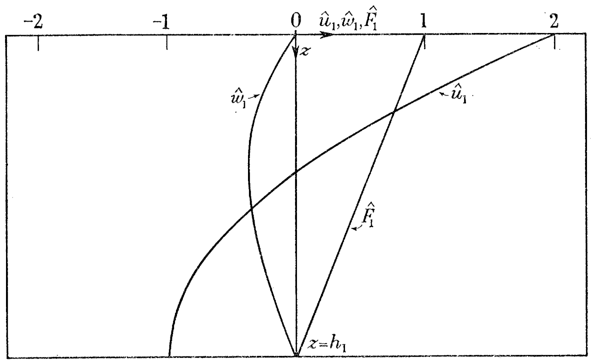

FIGURE 4. Variation with depth of the velocity components u_1, w_1, and the horizontal shear stress F_1, for steady circulatory motion in the top layer:

$$u_1 = \hat{u}_1 h_1 A(x)/6\rho_1 N_1,\quad w_1 = \hat{w}_1 h_1^2 A'(x)/6\rho_1 N_1,\quad F_1 = \hat{F}_1 A(x).$$

Integration yields the equation of a streamline in the form

$$(h_1-z)\{(h_1-z)^2-h_1^2\}A(x) = \text{constant}.\tag{74}$$

The steady motion given by (73) and (74) consists of at least one large-scale circulation in the top layer: streamlines corresponding to two different distributions, $A(x)$, of the wind stress F_S, are given in figure 3. Stagnant points are located where $A'(x)=0, z=h_1(1-1/\surd 3)$. There is a horizontal current in the direction of the wind stress down to a depth $h_1(1-1/\surd 3)$ below the surface, and a horizontal return current in the remaining depth of the top layer. Vertical current is zero when $A'(x)=0$ (i.e. at values of x where the horizontal current is a maximum); is directed upwards where $A'(x)>0$ and downwards where $A'(x)<0$; is greatest at the depth $z=h_1(1-1/\surd 3)$, and zero at $z=0$, h_1. Associated with the circulatory motion in the top layer is a shear stress F_1 given by (66); the shear stress F_2 in the bottom layer is zero (figure 4). Summarizing, the preceding analysis shows that a sudden increase in the wind stress acting over the surface of a narrow two-layered lake produces (i) ordinary and internal seiche oscillations, of the kind described by Proudman (1953, pp. 336 to 341); (ii) steady displacements of the water surface and the surface of discontinuity between the layers (equation (72)); (iii) large-scale circulatory motion in the top layer (equation (73)). This is the result obtained when internal friction at the surface of discontinuity and bottom friction are assumed to be zero.

5. Effects of bottom friction

The problem of determining the response of the two-layered lake to an instantaneous rise in wind-stress level at time $t = 0$, investigated in the last section, is again considered. Bottom friction, ignored previously, is now taken into account, and formulae are derived for the elevations, ζ_1, ζ_2, and the current u_1, in the motion of the lake. Instead of solving the differential equations (24) to (27) involving the current at any depth, the much simpler equations (19) to (22) are solved, again taking the internal friction at the surface of discontinuity to be zero ($F_D = 0$), but now making the approximation, similar to that used by Proudman (1954), in which the friction at the lake bottom is given by

$$\frac{F_B}{\rho_2 h_2} = 2ku_{2m}, \tag{33}$$

where k is some constant. With

$$F_S = H(t)\, A_n \sin(n\pi x/l), \tag{34}$$

and

$$\left.\begin{aligned} \zeta_1 &= Z_1(t)\cos(n\pi x/l), \quad \zeta_2 = Z_2(t)\cos(n\pi x/l), \\ u_{1m} &= U_{1m}(t)\sin(n\pi x/l), \quad u_{2m} = U_{2m}(t)\sin(n\pi x/l), \end{aligned}\right\} \tag{75}$$

where $u_{1m} = u_{2m} = 0$ at $x = 0$, $x = l$ as required by (28), (19) to (22) become

$$\left.\begin{aligned} \frac{\mathrm{d}U_{1m}}{\mathrm{d}t} &= \frac{n\pi g}{l} Z_1 + \frac{H(t)\, A_n}{\rho_1 h_1}, \\ \frac{\mathrm{d}U_{2m}}{\mathrm{d}t} + 2kU_{2m} &= \frac{n\pi g}{l}\frac{\rho_1}{\rho_2} Z_1 + \frac{n\pi g}{l}\left(1 - \frac{\rho_1}{\rho_2}\right) Z_2, \\ \frac{n\pi h_1}{l} U_{1m} + \frac{\mathrm{d}Z_1}{\mathrm{d}t} - \frac{\mathrm{d}Z_2}{\mathrm{d}t} &= 0, \\ \frac{n\pi h_2}{l} U_{2m} + \frac{\mathrm{d}Z_2}{\mathrm{d}t} &= 0. \end{aligned}\right\} \tag{76}$$

If we suppose that the water in the lake lies at rest initially, so that $U_{1m} = U_{2m} = Z_1 = Z_2 = 0$ when $t = 0$, taking Laplace transforms in (76) yields

$$\left.\begin{aligned} s\bar{U}_{1m} &= \frac{n\pi g}{l}\bar{Z}_1 + \left(\frac{A_n}{\rho_1 h_1}\right)\frac{1}{s}, \\ (s+2k)\,\bar{U}_{2m} &= \frac{n\pi g}{l}\frac{\rho_1}{\rho_2}\bar{Z}_1 + \frac{n\pi g}{l}\left(1 - \frac{\rho_1}{\rho_2}\right)\bar{Z}_2, \\ \frac{n\pi h_1}{l}\bar{U}_{1m} + s\bar{Z}_1 - s\bar{Z}_2 &= 0, \\ \frac{n\pi h_2}{l}\bar{U}_{2m} + s\bar{Z}_2 &= 0. \end{aligned}\right\} \tag{77}$$

The solution of these equations is

$$\left.\begin{aligned}\bar{Z}_1 &= -\left(\frac{lA_n}{n\pi\rho_1 h_1 g}\right)\frac{1}{s}+\frac{lA_n}{n\pi\rho_1 h_1 g}\left(s^3+2ks^2+\frac{n^2\pi^2 gh_2}{l^2}s\right)\Big/\chi(s),\\ \bar{Z}_2 &= \left(\frac{lA_n}{n\pi(\rho_2-\rho_1)\,h_1 g}\right)\frac{1}{s}-\frac{lA_n}{n\pi(\rho_2-\rho_1)\,h_1 g}\left(s^3+2ks^2+\frac{n^2\pi^2 gh}{l^2}s+2k\frac{n^2\pi^2 gh_1}{l^2}\right)\Big/\chi(s),\\ \bar{U}_{1m} &= \frac{A_n}{\rho_1 h_1}\left(s^2+2ks+\frac{n^2\pi^2 gh_2}{l^2}\right)\Big/\chi(s),\\ \bar{U}_{2m} &= -\frac{n^2\pi^2 gA_n}{l^2\rho_2}\Big/\chi(s),\end{aligned}\right\} \tag{78}$$

where $$\chi(s) = s^4+2ks^3+\frac{n^2\pi^2 gh}{l^2}s^2+\frac{2kn^2\pi^2 gh_1}{l^2}s+\frac{n^4\pi^4 g^2 h_1 h_2}{l^4}\left(1-\frac{\rho_1}{\rho_2}\right). \tag{79}$$

Considering cases in which $$\epsilon_n = \frac{4k^2 l^2 h_1 h_2}{n^2\pi^2 gh^3} \tag{80}$$

and $1-\rho_1/\rho_2$ are sufficiently small to be neglected in comparison with unity, we find that, approximately,

$$\chi(s) = \left\{\left(s+\frac{kh_2}{h}\right)^2+\frac{n^2\pi^2\sigma_{1,n}^2}{l^2}\right\}\left\{\left(s+\frac{kh_1}{h}\right)^2+\frac{n^2\pi^2\sigma_{2,n}^2}{l^2}\right\}, \tag{81}$$

where $$\left.\begin{aligned}\sigma_{1,n}^2 &= gh(1-\delta_{1,n}), & \delta_{1,n} &= \epsilon_n h_2/4h_1;\\ \sigma_{2,n}^2 &= \frac{gh_1 h_2}{h}\left(1-\frac{\rho_1}{\rho_2}\right)(1-\delta_{2,n}), & \delta_{2,n} &= \frac{\epsilon_n h^2}{4h_2^2}\Big/\left(1-\frac{\rho_1}{\rho_2}\right).\end{aligned}\right\} \tag{82}$$

With $\chi(s)$ given by (81), if we make further approximations in which ϵ_n, $h_1\epsilon_n/h_2$, $h_2\epsilon_n/h_1$, and $1-\rho_1/\rho_2$ are neglected in comparison with unity, the forms (78) may be expressed as follows:

$$\left.\begin{aligned}\bar{Z}_1 &= \frac{lAn}{n\pi\rho_1 h_1 g}\left\{-\frac{1}{s}+\frac{(h_1/h)\,(s+kh_2/h)+3kh_1 h_2/h^2}{(s+kh_2/h)^2+n^2\pi^2\sigma_{1,n}^2/l^2}+\frac{(h_2/h)\,(s+kh_1/h)-kh_1 h_2/h^2}{(s+kh_1/h)^2+n^2\pi^2\sigma_{2,n}^2/l^2}\right\},\\ \bar{Z}_2 &= \frac{lA_n}{n\pi(\rho_2-\rho_1)\,h_1 g}\left\{\frac{1}{s}-\frac{(s+kh_1/h)+kh_1/h}{(s+kh_1/h)^2+n^2\pi^2\sigma_{2,n}^2/l^2}\right\},\\ \bar{U}_{1m} &= \frac{A_n}{\rho_1 h_1}\left\{\frac{h_1/h-(\epsilon_n/k)\,(s+kh_2/h)}{(s+kh_2/h)^2+n^2\pi^2\sigma_{1,n}^2/l^2}+\frac{h_2/h+(\epsilon_n/k)(s+kh_1/h)}{(s+kh_1/h)^2+n^2\pi^2\sigma_{2,n}^2/l^2}\right\},\\ \bar{U}_{2m} &= \frac{A_n}{\rho_2 h}\left\{\frac{1-[\epsilon_n h(h_2-h_1)/2kh_1 h_2]\,(s+kh_2/h)}{(s+kh_2/h)^2+n^2\pi^2\sigma_{1,n}^2/l^2}+\frac{-1+[\epsilon_n h(h_2-h_1)/2kh_1 h_2]\,(s+kh_1/h)}{(s+kh_1/h)^2+n^2\pi^2\sigma_{2,n}^2/l^2}\right\}.\end{aligned}\right\} \tag{83}$$

Subsequently, Laplace transform inversion yields Z_1, Z_2, U_{1m}, U_{2m}, and these in (75) give

$$\left.\begin{aligned}\zeta_1 &= Z_1^{(n)}\cos(n\pi x/l), & \zeta_2 &= Z_2^{(n)}\cos(n\pi x/l),\\ u_{1m} &= U_{1m}^{(n)}\sin(n\pi x/l), & u_{2m} &= U_{2m}^{(n)}\sin(n\pi x/l),\end{aligned}\right\} \tag{84}$$

where

$$\left.\begin{aligned}
Z_1^{(n)} &= \frac{lA_n}{n\pi\rho_1 h_1 g}\left\{\begin{aligned}&-1+\left(\frac{h_1}{h}\cos\frac{n\pi\sigma_{1,n}t}{l}+\frac{3klh_1h_2}{n\pi h^2\sigma_{1,n}}\sin\frac{n\pi\sigma_{1,n}t}{l}\right)\exp\left(-\frac{kh_2t}{h}\right)\\&+\left(\frac{h_2}{h}\cos\frac{n\pi\sigma_{2,n}t}{l}-\frac{klh_1h_2}{n\pi h^2\sigma_{2,n}}\sin\frac{n\pi\sigma_{2,n}t}{l}\right)\exp\left(-\frac{kh_1t}{h}\right)\end{aligned}\right\},\\
Z_2^{(n)} &= \frac{lA_n}{n\pi(\rho_2-\rho_1)h_1g}\left\{1-\left(\cos\frac{n\pi\sigma_{2,n}t}{l}+\frac{kh_1l}{n\pi h\sigma_{2,n}}\sin\frac{n\pi\sigma_{2,n}t}{l}\right)\exp\left(-\frac{kh_1t}{h}\right)\right\},\\
U_{1m}^{(n)} &= \frac{A_n}{\rho_1h_1}\left\{\begin{aligned}&\left(\frac{lh_1}{n\pi h\sigma_{1,n}}\sin\frac{n\pi\sigma_{1,n}t}{l}-\frac{\epsilon_n}{k}\cos\frac{n\pi\sigma_{1,n}t}{l}\right)\exp\left(-\frac{kh_2t}{h}\right)\\&+\left(\frac{lh_2}{n\pi h\sigma_{2,n}}\sin\frac{n\pi\sigma_{2,n}t}{l}+\frac{\epsilon_n}{k}\cos\frac{n\pi\sigma_{2,n}t}{l}\right)\exp\left(-\frac{kh_1t}{h}\right)\end{aligned}\right\},\\
U_{2m}^{(n)} &= \frac{A_n}{\rho_2h}\left\{\begin{aligned}&\left(\frac{l}{n\pi\sigma_{1,n}}\sin\frac{n\pi\sigma_{1,n}t}{l}-\frac{\epsilon_nh(h_2-h_1)}{2kh_1h_2}\cos\frac{n\pi\sigma_{1,n}t}{l}\right)\exp\left(-\frac{kh_2t}{h}\right)\\&+\left(-\frac{l}{n\pi\sigma_{2,n}}\sin\frac{n\pi\sigma_{2,n}t}{l}+\frac{\epsilon_nh(h_2-h_1)}{2kh_1h_2}\cos\frac{n\pi\sigma_{2,n}t}{l}\right)\exp\left(-\frac{kh_1t}{h}\right)\end{aligned}\right\}.
\end{aligned}\right\} \tag{85}$$

The elevations and mean currents given by (84) are associated with a wind stress

$$F_S = H(t)\,A_n\sin(n\pi x/l).$$

Arguing as in §4, it follows that, corresponding to

$$F_S = H(t)\,A(x), \tag{60}$$

where

$$A(x) = \sum_{n=1}^{\infty} A_n\sin(n\pi x/l), \tag{61}$$

we have

$$\left.\begin{aligned}\zeta_1 &= \sum_{n=1}^{\infty} Z_1^{(n)}\cos(n\pi x/l), & \zeta_2 &= \sum_{n=1}^{\infty} Z_2^{(n)}\cos(n\pi x/l),\\ u_{1m} &= \sum_{n=1}^{\infty} U_{1m}^{(n)}\sin(n\pi x/l), & u_{2m} &= \sum_{n=1}^{\infty} U_{2m}^{(n)}\sin(n\pi x/l).\end{aligned}\right\} \tag{86}$$

The forms (85) and (86) indicate the presence of damped seiches of periods $2l/n\sigma_{1,n}$, $2l/n\sigma_{2,n}$ ($n = 1, 2, 3 \ldots$) in the lake as a result of the sudden rise in wind stress given by (60). The case of zero bottom friction is obtained by putting $k = 0$: ζ_1, ζ_2, u_{1m}, u_{2m} given by (86) then reduce, as expected, to ζ_1, ζ_2, u_{1m}, u_{2m} deducible from (63) and (69). The period $2l/n\sigma_{1,n}$ is associated with a surface seiche of wavelength $2l/n$, for as $k \to 0$: $2l/n\sigma_{1,n} \to 2l/n(gh)^{\frac{1}{2}}$, which is the period of the surface seiche of wavelength $2l/n$ in the case of zero bottom friction (equation (70)). Similarly, the period $2l/n\sigma_{2,n}$ is associated with an internal seiche of wavelength $2l/n$. The periods $2l/n\sigma_{1,n}$, $2l/n\sigma_{2,n}$ ($n = 1, 2, 3, \ldots$) of the surface and internal seiches, respectively, depend upon the frictional coefficient k according to the relations (82). Since we have considered $\epsilon_n \ll 1$, it follows that $\sigma_{1,n}^2 \sim gh$, i.e. the friction has a negligible effect on the periods of the surface seiches. Its effect on the periods of the internal seiches is negligible when $\delta_{2,n} \ll 1$, for then

$$\sigma_{2,n}^2 \sim \frac{gh_1h_2}{h}\left(1-\frac{\rho_1}{\rho_2}\right).$$

The Laplace transform inversion yielding $Z_1^{(n)}$, $Z_2^{(n)}$, $U_{1m}^{(n)}$, $U_{2m}^{(n)}$ given by (85), is appropriate when

$$k^2 < n^2\pi^2 ghh_2(\rho_2-\rho_1)/\rho_2 l^2 h_1,$$

i.e. when $\delta_{2,n} < 1$, $\sigma_{2,n}^2 > 0$. However, when

$$k^2 > n^2\pi^2 ghh_2(\rho_2-\rho_1)/\rho_2 l^2 h_1,$$

then $\delta_{2,n} > 1$ and $\sigma_{2,n}^2 = -\sigma_{2,n}'^2 < 0$, in which case $Z_1^{(n)}$, $Z_2^{(n)}$, $U_{1m}^{(n)}$, $U_{2m}^{(n)}$, derived from the inversion are given by (85) with

$$\left.\begin{matrix}\sin\\ \cos\end{matrix}\right\}\frac{n\pi\sigma_{2,n}t}{l} \quad \text{replaced by} \quad \left.\begin{matrix}\sinh\\ \cosh\end{matrix}\right\}\frac{n\pi\sigma_{2,n}'t}{l},$$

and $\sigma_{2,n}$ by $\sigma_{2,n}'$. Bottom friction is then great enough to prevent the development of the internal seiche oscillation of wavelength $2l/n$. When

$$k^2 = n^2\pi^2 ghh_2(\rho_2-\rho_1)/\rho_2 l^2 h_1,$$

then $\delta_{2,n} = 1$, $\sigma_{2,n}^2 = 0$; in this case the appropriate forms for $Z_1^{(n)}$, $Z_2^{(n)}$, $U_{1m}^{(n)}$, $U_{2m}^{(n)}$ are obtained from (85) by replacing $(l/n\pi\sigma_{2,n})\sin(n\pi\sigma_{2,n}t/l)$ by t, and $\cos(n\pi\sigma_{2,n}t/l)$ by unity. The friction is then just great enough to prevent the development of the internal oscillation of wavelength $2l/n$.

Internal seiches centred on the thermocline boundary may be regarded as the normal response of a stratified lake to the wind (Mortimer 1953). In general, therefore, according to the above argument, we expect $k^2 < \pi^2 ghh_2(\rho_2-\rho_1)/\rho_2 l^2 h_1$ in a two-layered model. There is, however, evidence from Windermere northern basin (Mortimer 1955) to show that just before the autumn overturn, at times of low stability when $1-\rho_1/\rho_2$ is comparatively small and h_1 comparatively large, the wind does not set up any conspicuous internal oscillation in the Lake—suggesting that in a two-layered model the condition $k^2 \geqslant \pi^2 ghh_2(\rho_2-\rho_1)/\rho_2 l^2 h_1$ might then apply.

With u_{1m} known from (86), the horizontal current at any depth in the top layer, u_1, may be determined from

$$u_1 = u_{1m} + \frac{A(x)}{6\rho_1 N_1 h_1}\{3(h_1-z)^2 - h_1^2\} + \frac{2h_1 A(x)}{\rho_1 N_1 \pi^2}\sum_{r=1}^{\infty}\frac{(-1)^{r+1}}{r^2}\exp\left(-\frac{r^2\pi^2 N_1 t}{h_1^2}\right)\cos\frac{r\pi}{h_1}(h_1-z). \quad (87)$$

This relation is suggested by the third equation of (63). It may be verified that u_1 thus given satisfies the differential equations (24) and (26), the boundary conditions (28) to (30), and the initial condition: $u_1 = 0$ when $t = 0$.

An examination of (87) shows that u_1 consists of a part u_{1m}, independent of the depth z, due to the seiche motion in the lake, and a part due to the wind-driven circulation discussed in §4. The horizontal current at any depth in the bottom layer, u_2, cannot be determined from the theory of the present section; this current will vary with the depth under the shearing effect of bottom friction. The shearing does not reach up into the top layer because of the assumption of zero internal friction at the surface of discontinuity.

6. Solution of the equations with zero bottom current

Again, as in § 4, we seek a solution of equations (24) to (27) to determine the motion in the two-layered lake resulting from a sudden increase in the wind stress F_S at time $t = 0$. The condition that the bottom friction is zero, employed in § 4, is now replaced by the condition that the bottom current is zero; equations (28) to (31) give the complete set of boundary conditions. The analysis is more general than that given in either § 4 or § 5, for bottom friction is taken into account, and the current at any depth in *either* layer may be determined.

Equations (34) to (43) are applicable with the last condition of (43) replaced by

$$\bar{U}_2 = 0 \quad \text{at} \quad z = h. \tag{88}$$

Substituting $\bar{U}_1$, $\bar{U}_2$ given by (41) into (39), (40) and (43) thus modified, yields

$$\left(\frac{n^2\pi^2 gh_1}{l^2}+s^2\right)\bar{Z}_1 - s^2\bar{Z}_2 = \frac{n\pi s}{l\alpha_1}[-B_1\sinh\alpha_1 h_1 + C_1(1-\cosh\alpha_1 h_1)], \tag{89}$$

$$\frac{n^2\pi^2 gh_2}{l^2}\frac{\rho_1}{\rho_2}\bar{Z}_1 + \left[\frac{n^2\pi^2 gh_2}{l^2}\left(1-\frac{\rho_1}{\rho_2}\right)+s^2\right]\bar{Z}_2$$

$$= \frac{n\pi s}{l\alpha_2}[B_2(\sinh\alpha_2 h_1 - \sinh\alpha_2 h) + C_2(\cosh\alpha_2 h_1 - \cosh\alpha_2 h)], \tag{90}$$

$$-\rho_1 N_1 \alpha_1 C_1 = A_n/s, \tag{91}$$

$$B_1\sinh\alpha_1 h_1 + C_1\cosh\alpha_1 h_1 = 0, \tag{92}$$

$$B_2\sinh\alpha_2 h_1 + C_2\cosh\alpha_2 h_1 = 0, \tag{93}$$

$$B_2\cosh\alpha_2 h + C_2\sinh\alpha_2 h + \frac{n\pi g}{ls}\frac{\rho_1}{\rho_2}\bar{Z}_1 + \frac{n\pi g}{ls}\left(1-\frac{\rho_1}{\rho_2}\right)\bar{Z}_2 = 0. \tag{94}$$

Then, solving these equations for $\bar{Z}_1$, $\bar{Z}_2$, $\bar{U}_1$, $\bar{U}_2$, we get

$$\left.\begin{aligned}
\bar{Z}_1 &= -\left(\frac{lA_n}{n\pi\rho_1 h_1 g}\right)\frac{1}{s} + \frac{lA_n}{n\pi\rho_1 h_1 g}\left[s^3 + \frac{n^2\pi^2 gs}{l^2}\left(h_2 - \frac{\tanh\alpha_2 h_2}{\alpha_2}\right)\right]\frac{1}{\psi(s)},\\
\bar{Z}_2 &= \left(\frac{lA_n}{n\pi(\rho_2-\rho_1)h_1 g}\right)\frac{1}{s} - \frac{lA_n}{n\pi(\rho_2-\rho_1)h_1 g}\left[s^3 + \frac{n^2\pi^2 gs}{l^2}\left(h_1 + h_2 - \frac{\tanh\alpha_2 h_2}{\alpha_2}\right)\right]\frac{1}{\psi(s)},\\
\bar{U}_1 &= -\left(\frac{A_n}{\rho_1 h_1}\right)\frac{1}{s^2} + \frac{A_n\cosh\{\alpha_1(h_1-z)\}}{s\rho_1 N_1\alpha_1\sinh\alpha_1 h_1} + \frac{A_n}{\rho_1 h_1}\left[s^2 + \frac{n^2\pi^2 g}{l^2}\left(h_2 - \frac{\tanh\alpha_2 h_2}{\alpha_2}\right)\right]\frac{1}{\psi(s)},\\
\bar{U}_2 &= -\frac{n^2\pi^2 gA_n}{l^2\rho_2}\left[1 - \frac{\cosh\{\alpha_2(z-h_1)\}}{\cosh\alpha_2 h_2}\right]\frac{1}{\psi(s)},
\end{aligned}\right\} \tag{95}$$

where
$$\psi(s) = \Phi(s) - \frac{n^2\pi^2 g}{l^2}\frac{\tanh\alpha_2 h_2}{\alpha_2}\left\{s^2 + \frac{n^2\pi^2 gh_1}{l^2}\left(1-\frac{\rho_1}{\rho_2}\right)\right\}. \tag{96}$$

The function $\Phi(s)$ is defined by (51).

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Laplace inversion in (95), using (57), yields

$$\left.\begin{aligned} Z_1 &= \frac{lA_n}{n\pi\rho_1 h_1 g}(-1+J_1), \\ Z_2 &= \frac{lA_n}{n\pi(\rho_2-\rho_1)\,h_1 g}(1-J_2), \\ U_1 &= \frac{A_n}{6\rho_1 N_1 h_1}\{3(h_1-z)^2-h_1^2\} \\ &\quad +\frac{2h_1 A_n}{\rho_1 N_1 \pi^2}\sum_{r=1}^{\infty}\frac{(-1)^{r+1}}{r^2}\exp\left(-\frac{r^2\pi^2 N_1 t}{h_1^2}\right)\cos\left\{\frac{r\pi}{h_1}(h_1-z)\right\}-\frac{h_2^2 A_n}{\rho_1 h_1 N_2}J_3, \\ U_2 &= \frac{h_2 A_n}{\rho_2 N_2}J_4, \end{aligned}\right\} \tag{97}$$

where, from the inversion theorem,

$$\left.\begin{aligned} J_1 &= \frac{1}{2\pi \mathrm{i}}\int_{\gamma-\mathrm{i}\infty}^{\gamma+\mathrm{i}\infty}\left[\lambda^3+\frac{n^2\pi^2 g\lambda}{l^2}\left(h_2-\frac{\tanh\{h_2(\lambda/N_2)^{\frac{1}{2}}\}}{(\lambda/N_2)^{\frac{1}{2}}}\right)\right]\frac{\exp(\lambda t)}{\psi(\lambda)}\,\mathrm{d}\lambda, \\ J_2 &= \frac{1}{2\pi \mathrm{i}}\int_{\gamma-\mathrm{i}\infty}^{\gamma+\mathrm{i}\infty}\left[\lambda^3+\frac{n^2\pi^2 g\lambda}{l^2}\left(h_1+h_2-\frac{\tanh\{h_2(\lambda/N_2)^{\frac{1}{2}}\}}{(\lambda/N_2)^{\frac{1}{2}}}\right)\right]\frac{\exp(\lambda t)}{\psi(\lambda)}\,\mathrm{d}\lambda, \\ J_3 &= -\frac{1}{2\pi \mathrm{i}}\int_{\gamma-\mathrm{i}\infty}^{\gamma+\mathrm{i}\infty}\frac{N_2}{h_2^2}\left[\lambda^2+\frac{n^2\pi^2 g}{l^2}\left(h_2-\frac{\tanh\{h_2(\lambda/N_2)^{\frac{1}{2}}\}}{(\lambda/N_2)^{\frac{1}{2}}}\right)\right]\frac{\exp(\lambda t)}{\psi(\lambda)}\,\mathrm{d}\lambda, \\ J_4 &= -\frac{1}{2\pi \mathrm{i}}\int_{\gamma-\mathrm{i}\infty}^{\gamma+\mathrm{i}\infty}\frac{n^2\pi^2 g N_2}{l^2 h_2}\left[1-\frac{\cosh\{(z-h_1)\,(\lambda/N_2)^{\frac{1}{2}}\}}{\cosh\{h_2(\lambda/N_2)^{\frac{1}{2}}\}}\right]\frac{\exp(\lambda t)}{\psi(\lambda)}\,\mathrm{d}\lambda. \end{aligned}\right\} \tag{98}$$

The problem now is to evaluate these integrals by contour integration. The singularities of each integrand coincide with the non-zero roots of $\psi(\lambda) = 0$. This equation may be written in the non-dimensonal form:

$$\tan\nu = \frac{a\nu^9+b\nu^5+c\nu}{\nu^4+c}, \tag{99}$$

where

$$\nu = \mathrm{i}h_2(\lambda/N_2)^{\frac{1}{2}}, \tag{100}$$

$$a = l^2 N_2^2/n^2\pi^2 g h_2^5, \tag{101}$$

$$b = 1+h_1/h_2, \tag{102}$$

$$c = d/a, \tag{103}$$

$$d = h_1(1-\rho_1/\rho_2)/h_2. \tag{104}$$

Here, a, b, d are the basic parameters: for any particular lake b, d are easily fixed, but a may be regarded as variable since N_2 is, in general, unknown. No distinction is drawn between roots of (99) which differ only in sign, for these correspond to the same value of λ. Using (99) to express a as a function of ν, for $0 < \nu < \frac{1}{2}\pi$, we get

$$a = [\tan\nu - b\nu + \{(\tan\nu - b\nu)^2 + 4d\nu(\tan\nu - \nu)\}^{\frac{1}{2}}]/2\nu^5, \tag{105}$$

from which it follows that a has a minimum value, a_0, say, at some point $\nu = \nu_0$ $(0 < \nu_0 < \frac{1}{2}\pi)$. Hence, if $a < a_0$ there are no real roots of the equation (99) for $0 < \nu < \frac{1}{2}\pi$. The equation then has two pairs of conjugate complex roots. One pair, denoted by $\kappa_1 \pm \mathrm{i}\tau_1$, corresponds to the roots $\lambda = \pm \mathrm{i}n\pi\beta_1/l$ of the equation $\Phi(\lambda) = 0$; the other pair, denoted by $\kappa_2 \pm \mathrm{i}\tau_2$, corresponds to the roots $\lambda = \pm in\pi\beta_2/l$ of $\Phi(\lambda) = 0$. If $a = a_0$, (99) has one real root $\nu = \nu_0$ for $0 < \nu < \frac{1}{2}\pi$, and one pair of conjugate complex roots $\kappa_1 \pm i\tau_1$. If $a > a_0$, the equation has two different real roots in the range $0 < \nu < \frac{1}{2}\pi$ denoted by ν_{01}, ν_{02}, and again a pair of conjugate complex roots $\kappa_1 \pm i\tau_1$. For every value of a real roots exist which are approximately equal to $\frac{1}{2}r\pi$ $(r = 3, 5, 7, \ldots)$, denoted by $\nu_1, \nu_2, \nu_3, \ldots$.

If we use (100) in the form

$$\lambda = -N_2\nu^2/h_2^2, \tag{106}$$

it follows that, apart from zero, the roots of $\psi(\lambda) = 0$ are:

for $a < a_0$, $\quad -N_2(\kappa_1^2-\tau_1^2 \pm 2\kappa_1\tau_1 i)/h_2^2, \quad -N_2(\kappa_2^2-\tau_2^2 \pm 2\kappa_2\tau_2 i)/h_2^2$;

for $a = a_0$, $\quad -N_2(\kappa_1^2-\tau_1^2 \pm 2\kappa_1\tau_1 i)/h_2^2, \quad -N_2\nu_0^2/h_2^2$;

for $a > a_0$, $\quad -N_2(\kappa_1^2-\tau_1^2 \pm 2\kappa_1\tau_1 i)/h_2^2, \quad -N_2\nu_{01}^2/h_2^2, \quad -N_2\nu_{02}^2/h_2^2$;

and for all a, $\quad -N_2\nu_1^2/h_2^2, \quad -N_2\nu_2^2/h_2^2, \quad -N_2\nu_3^2/h_2^2, \ldots$;

where $\kappa_1, \tau_1, \kappa_2, \tau_2, \nu_{01}, \nu_{02}, \nu_r (r = 1, 2, 3, \ldots)$ are functions of a, b, and d; and a_0, ν_0 functions of b and d. The above roots are simple poles of the integrands of J_1, J_2, J_3 and J_4, given by (98), and these integrals may now be evaluated using the calculus of residues. A contour similar to that used for the evaluation of I in §4 is taken. Then, using Cauchy's theorem as before, we deduce that each integral equals $2\pi i \times$ (sum of the residues at the poles of its integrand). Thus

$$\left.\begin{aligned}
J_1 &= \sum_\nu \left[\lambda\left\{\lambda^2+\frac{n^2\pi^2 g}{l^2}\left(h_2-\frac{\tanh\{h_2(\lambda/N_2)^{\frac{1}{2}}\}}{(\lambda/N_2)^{\frac{1}{2}}}\right)\right\}\frac{\exp(\lambda t)}{\psi'(\lambda)}\right]_{\lambda=-N_2\nu^2/h_2^2},\\
J_2 &= \sum_\nu \left[\lambda\left\{\lambda^2+\frac{n^2\pi^2 g}{l^2}\left(h_1+h_2-\frac{\tanh\{h_2(\lambda/N_2)^{\frac{1}{2}}\}}{(\lambda/N_2)^{\frac{1}{2}}}\right)\right\}\frac{\exp(\lambda t)}{\psi'(\lambda)}\right]_{\lambda=-N_2\nu^2/h_2^2},\\
J_3 &= \sum_\nu -\frac{N_2}{h_2^2}\left[\left\{\lambda^2+\frac{n^2\pi^2 g}{l^2}\left(h_2-\frac{\tanh\{h_2(\lambda/N_2)^{\frac{1}{2}}\}}{(\lambda/N_2)^{\frac{1}{2}}}\right)\right\}\frac{\exp(\lambda t)}{\psi'(\lambda)}\right]_{\lambda=-N_2\nu^2/h_2^2},\\
J_4 &= \sum_\nu -\frac{n^2\pi^2 g N_2}{l^2 h_2}\left[\left\{1-\frac{\cosh\{(z-h_1)(\lambda/N_2)^{\frac{1}{2}}\}}{\cosh\{h_2(\lambda/N_2)^{\frac{1}{2}}\}}\right\}\frac{\exp(\lambda t)}{\psi'(\lambda)}\right]_{\lambda=-N_2\nu^2/h_2^2},
\end{aligned}\right\}$$

which reduce to

$$J_q = \sum_\nu H_q(\nu)\exp(-N_2\nu^2 t/h_2^2) \quad (q = 1, 2, 3, 4), \tag{107}$$

where

$$\left.\begin{aligned}
H_1 &= 2(a\nu^8+\nu^4-\nu^3\tan\nu)/E,\\
H_2 &= 2(a\nu^8+b\nu^4-\nu^3\tan\nu)/E,\\
H_3 &= 2(a\nu^6+\nu^2-\nu\tan\nu)/E,\\
H_4 &= 2\nu^2[1-\cos(\xi\nu)/\cos\nu]/E,\\
E &= 9a\nu^8+(5b-1)\nu^4-4\nu^3\tan\nu-(\nu^4+c)\tan^2\nu,\\
\xi &= (z-h_1)/h_2,
\end{aligned}\right\} \tag{108}$$

each summation extending over the various non-zero roots of (99) corresponding to different values of λ. In the usual notation $\psi'(\lambda) = d\psi/d\lambda$. Now using (99), it may be shown that, in (107),

$$H_q(\kappa+i\tau) = L_q(\kappa,\tau)+iM_q(\kappa,\tau), \tag{109}$$

where

$$\left.\begin{aligned}
L_1 &= 2(ac+1-b)\,(l_1 e_1+m_1 e_2)/e,\\
M_1 &= 2(ac+1-b)\,(m_1 e_1-l_1 e_2)/e,\\
L_2 &= 2c(l_2 e_1+m_2 e_2)/e,\\
M_2 &= 2c(m_2 e_1-l_2 e_2)/e,\\
L_3 &= 2(ac+1-b)\,(l_3 e_1+m_3 e_2)/e,\\
M_3 &= 2(ac+1-b)\,(m_3 e_1-l_3 e_2)/e,\\
L_4 &= 2\{(1-l_5)\,(l_4 e_1+m_4 e_2)+m_5(m_4 e_1-l_4 e_3)\}/e,\\
M_4 &= 2\{(1-l_5)\,(m_4 e_1-l_4 e_2)-m_5(l_4 e_1+m_4 e_2)\}/e,
\end{aligned}\right\} \tag{110}$$

the newly introduced real quantities l_1, m_1, e_1, etc., being determined in terms of κ, τ (also real) from the relations

$$\left.\begin{aligned}
&l_1+\mathrm{i}m_1 = \sigma^6, \quad l_2+\mathrm{i}m_2 = a\sigma^6+(b-1)\,\sigma^2, \quad l_3+\mathrm{i}m_3 = \sigma^4, \quad l_4+\mathrm{i}m_4 = \sigma^4+c,\\
&l_5 = 2(\cos\xi\kappa\cosh\xi\tau\cos\kappa\cosh\tau+\sin\xi\kappa\sinh\xi\tau\sin\kappa\sinh\tau)/(\cos 2\kappa+\cosh 2\tau),\\
&m_5 = 2(\cos\xi\kappa\cosh\xi\tau\sin\kappa\sinh\tau-\sin\xi\kappa\sinh\xi\tau\cos\kappa\cosh\tau)/(\cos 2\kappa+\cosh 2\tau),\\
&e_1+\mathrm{i}e_2 = -a^2\sigma^{16}-2ab\sigma^{12}+5a\sigma^{10}-(b^2+2ac)\,\sigma^8+(9ac+b-1)\,\sigma^6-2bc\sigma^4\\
&\qquad\qquad +5(b-1)\,c\sigma^2-c^2,\\
&e = e_1^2+e_2^2,\\
&(\sigma = \kappa+\mathrm{i}\tau).
\end{aligned}\right\} \tag{111}$$

Hence, from (107) with values of ν inserted, we get

$$\left.\begin{aligned}
J_q &= G_q(\kappa_1,\tau_1)\exp(-\gamma_1 t)+G_q(\kappa_2,\tau_2)\exp(-\gamma_2 t)\\
&\qquad +\sum_{r=1}^{\infty} H_q(\nu_r)\exp(-N_2\nu_r^2 t/h_2^2) \quad (a<a_0),\\
J_q &= G_q(\kappa_1,\tau_1)\exp(-\gamma_1 t)+H_q(\nu_0)\exp(-N_2\nu_0^2 t/h_2^2)\\
&\qquad +\sum_{r=1}^{\infty} H_q(\nu_r)\exp(-N_2\nu_r^2 t/h_2^2) \quad (a=a_0),\\
J_q &= G_q(\kappa_1,\tau_1)\exp(-\gamma_1 t)+H_q(\nu_{01})\exp(-N_2\nu_{01}^2 t/h_2^2)+H_q(\nu_{02})\exp(-N_2\nu_{02}^2 t/h_2^2)\\
&\qquad +\sum_{r=1}^{\infty} H_q(\nu_r)\exp(-N_2\nu_r^2 t/h_2^2) \quad (a>a_0),
\end{aligned}\right\} \tag{112}$$

where
$$G_q(\kappa,\tau) = 2L_q(\kappa,\tau)\cos(2N_2\kappa\tau t/h_2^2)+2M_q(\kappa,\tau)\sin(2N_2\kappa\tau t/h_2^2), \tag{113}$$

and
$$\gamma_1 = N_2(\kappa_1^2-\tau_1^2)/h_2^2, \quad \gamma_2 = N_2(\kappa_2^2-\tau_2^2)/h_2^2. \tag{114}$$

Taking $q = 1, 2, 3, 4$ in (112) gives J_1, J_2, J_3 and J_4. These in (97), and then Z_1, Z_2, U_1, U_2 from (97) in (35), give ζ_1, ζ_2, u_1, u_2 in the response of the lake to the wind stress

$$F_S = H(t)\,A_n\sin(n\pi x/l).$$

Thus
$$\left.\begin{aligned}
\zeta_1 &= Z_1^{(n)}\cos(n\pi x/l), \quad \zeta_2 = Z_2^{(n)}\cos(n\pi x/l),\\
u_1 &= U_1^{(n)}\sin(n\pi x/l), \quad u_2 = U_2^{(n)}\sin(n\pi x/l),
\end{aligned}\right\} \tag{115}$$

where, in the case when $a < a_0$,

$$\left.\begin{aligned} Z_1^{(n)} &= \frac{lA_n}{n\pi\rho_1 h_1 g}\Big\{-1+G_1(\kappa_1,\tau_1)\exp(-\gamma_1 t)+G_1(\kappa_2,\tau_2)\exp(-\gamma_2 t) \\ &\qquad +\sum_{r=1}^{\infty} H_1(\nu_r)\exp(-N_2\nu_r^2 t/h_2^2)\Big\}, \\ Z_2^{(n)} &= \frac{lA_n}{n\pi(\rho_2-\rho_1)h_1 g}\Big\{1-G_2(\kappa_1,\tau_1)\exp(-\gamma_1 t)-G_2(\kappa_2,\tau_2)\exp(-\gamma_2 t) \\ &\qquad -\sum_{r=1}^{\infty} H_2(\nu_r)\exp(-N_2\nu_r^2 t/h_2^2)\Big\}, \\ U_1^{(n)} &= \frac{A_n}{6\rho_1 N_1 h_1}\{3(h_1-z)^2-h_1^2\}+\frac{2h_1 A_n}{\rho_1 N_1 \pi^2}\sum_{r=1}^{\infty}\frac{(-1)^{r+1}}{r^2}\exp\left(-\frac{r^2\pi^2 N_1 t}{h_1^2}\right)\cos\left\{\frac{r\pi}{h_1}(h_1-z)\right\} \\ &\qquad -\frac{h_2^2 A_n}{\rho_1 h_1 N_2}\Big\{G_3(\kappa_1,\tau_1)\exp(-\gamma_1 t)+G_3(\kappa_2,\tau_2)\exp(-\gamma_2 t) \\ &\qquad +\sum_{r=1}^{\infty} H_3(\nu_r)\exp(-N_2\nu_r^2 t/h_2^2)\Big\}, \\ U_2^{(n)} &= \frac{h_2 A_n}{\rho_2 N_2}\Big\{G_4(\kappa_1,\tau_1)\exp(-\gamma_1 t)+G_4(\kappa_2,\tau_2)\exp(-\gamma_2 t) \\ &\qquad +\sum_{r=1}^{\infty} H_4(\nu_r)\exp(-N_2\nu_r^2 t/h_2^2)\Big\}. \end{aligned}\right\} \quad (116)$$

Examine equations (115) and (116) and compare them with (59) of §4, and (84) and (85) of §5. Terms in $U_1^{(n)}$ representing circulatory motion in the top layer may again be recognized; also trigonometric terms in x and t, representing two seiches of wavelength $2l/n$, with periods

$$T_1 = \pi h_2^2/N_2\kappa_1\tau_1, \quad T_2 = \pi h_2^2/N_2\kappa_2\tau_2, \quad (117)$$

and damping factors $$f_1 = \exp(-\gamma_1 t), \quad f_2 = \exp(-\gamma_2 t), \quad (118)$$

respectively. When $a \geqslant a_0$, i.e. $N_2 \geqslant n\pi(ga_0h_2^5)^{\frac{1}{2}}/l$, (112) shows that there are no terms in $Z_1^{(n)}$, $Z_2^{(n)}$, $U_1^{(n)}$, $U_2^{(n)}$ corresponding to an oscillation of period T_2. Evidently friction is then great enough to prevent the development of this mode which is, therefore, an internal seiche, the mode of period T_1 being a surface seiche: thus interpreted the results obtained here are consistent with those obtained in §5, the condition $N_2 \geqslant n\pi(ga_0h_2^5)^{\frac{1}{2}}/l$ given above corresponding to the condition $k^2 \geqslant n^2\pi^2 ghh_2(\rho_2-\rho_1)/\rho_2 l^2 h_1$ derived in that section.

The response of the lake in the more general case when

$$F_S = H(t)\sum_{n=1}^{\infty} A_n \sin(n\pi x/l)$$

is, from (115),

$$\left.\begin{aligned} \zeta_1 &= \sum_{n=1}^{\infty} Z_1^{(n)}\cos(n\pi x/l), \quad & \zeta_2 &= \sum_{n=1}^{\infty} Z_2^{(n)}\cos(n\pi x/l), \\ u_1 &= \sum_{n=1}^{\infty} U_1^{(n)}\sin(n\pi x/l), \quad & u_2 &= \sum_{n=1}^{\infty} U_2^{(n)}\sin(n\pi x/l), \end{aligned}\right\} \quad (119)$$

where it should be noted that in $Z_1^{(n)}, Z_2^{(n)}, U_1^{(n)}, U_2^{(n)}$ there is a different set of values of κ_1, τ_1, κ_2, τ_2, ν_{01}, ν_{02} and ν_r $(r = 1, 2, 3, \ldots)$ for each different n. Seiches of different nodality are now present in the motion of the water. As indicated above, friction is great enough to prevent the development of the internal seiche of nodality n if

$$N_2 \geqslant n\pi(ga_0h_2^5)^{\frac{1}{2}}/l. \quad (120)$$

In a stratified lake, of all the internal waves which are theoretically possible the one most commonly set in motion is a uninodal seiche on the thermocline boundary (Mortimer 1953). Hence, corresponding to most natural conditions we expect $N_2 < \pi(ga_0h_2^5)^{\frac{1}{2}}/l$ in the two-layered model. This inequality corresponds to $k^2 < \pi^2ghh_2(\rho_2-\rho_1)/\rho_2l^2h_1$, i.e. $\epsilon_1 < 4h_2^2(1-\rho_1/\rho_2)/h^2$, given in § 5.

7. Response to a variable wind stress

In §§ 4, 5 and 6, the elevations ζ_1, ζ_2 and the currents u_1, u_2, u_{1m}, u_{2m} associated with a wind-stress distribution

$$F_S = H(t)\,A(x) \tag{60}$$

have been determined. Let these elevations and currents be denoted, respectively, by ζ_1', ζ_2', u_1', u_2', u_{1m}', u_{2m}'; each is a function of t and x, zero at $t = 0$. Then, the corresponding elevations and currents for a distribution

$$F_S = \Theta(t)\,A(x) \quad (t \geqslant 0), \tag{121}$$

where $\Theta(t)$ is some general function of t, are given by

$$\left.\begin{aligned} \zeta_1 &= \int_0^t \Theta(\tau)\,\frac{\partial \zeta_1'}{\partial t}\,(t-\tau, x)\,\mathrm{d}\tau,\\ \zeta_2 &= \int_0^t \Theta(\tau)\,\frac{\partial \zeta_2'}{\partial t}\,(t-\tau, x)\,\mathrm{d}\tau,\\ u_1 &= \int_0^t \Theta(\tau)\,\frac{\partial u_1'}{\partial t}\,(t-\tau, x)\,\mathrm{d}\tau, \end{aligned}\right\} \tag{122}$$

with similar integrals for u_2, u_{1m}, u_{2m}. Integrals of this type have been formulated by Proudman & Doodson (1924, p. 146). In particular, when $\Theta(t)$ consists of a series of steps in time defined by

$$\left.\begin{aligned} \Theta(t) &= \Theta_1 \quad (0 \leqslant t \leqslant t_1),\\ &= \Theta_2 \quad (t_1 < t \leqslant t_2),\\ &= \Theta_3 \quad (t_2 < t \leqslant t_3),\\ &\quad \text{etc.,} \end{aligned}\right\} \tag{123}$$

where $\Theta_1, \Theta_2, \ldots, \quad t_1, t_2, \ldots$

are constants, then

$$\left.\begin{aligned} \zeta_1 &= \Theta_1\zeta_1'(t, x) \quad (0 \leqslant t \leqslant t_1),\\ &= \Theta_1\zeta_1'(t, x) + (\Theta_2-\Theta_1)\,\zeta_1'(t-t_1, x) \quad (t_1 < t \leqslant t_2),\\ &= \Theta_1\zeta_1'(t, x) + (\Theta_2-\Theta_1)\,\zeta_1'(t-t_1, x) + (\Theta_3-\Theta_2)\,\zeta_1'(t-t_2, x) \quad (t_2 < t \leqslant t_3),\\ &\quad \text{etc.,} \end{aligned}\right\} \tag{124}$$

with similar expressions for ζ_2, u_1, u_2, u_{1m}, u_{2m}. In the application of the present theory to Windermere, described in part III, a time-variation of wind stress of the form given by (123) is used in representing the actual variation over the lake deduced from anemometer records.

PART II. ANALYSIS OF OBSERVATIONS FROM WINDERMERE

By N. S. Heaps and A. E. Ramsbottom

8. Introduction

In part I of this paper a theory has been developed for the motion of a narrow two-layered lake under wind stress. The theory is applied, in part III, to determine fluctuations in the level of the thermocline, and mean currents, in Windermere northern basin during a period of summer stratification. Observational data for this period, relating to Windermere, are described and discussed in §§ 9, 10 and 11 which follow.

Temperature observations in the lake, taken by Mortimer (1953), have been analysed to determine the vertical motion of the thermocline at one station; wind measurements have been used to estimate the varying component of the wind stress directed along the length of the lake. The results of these analyses are now presented and examined.

The processing of the data given by temperature and wind records has been carried out mainly by one of us (A. E. R.) at the Windermere laboratory of the Freshwater Biological Association. Responsibility for the methods used, and the interpretation of the results obtained, rests with the first author.

9. Analysis of temperature records

A chart of the northern basin of Windermere, showing the orientation of the lake, its depth contours, and the medial line through the deepest points is given in figure 5. The vertical longitudinal section, taken to be the depth variation along the medial line, is shown in figure 6.

Temperatures of the water in Windermere northern basin have been measured by Mortimer (1953) at nine fixed depths spaced throughout the water column at a station *S*—marked in figures 5 and 6. The measurements were carried out by means of electrical resistance thermometers moored in the lake and connected to recorders ashore. These were arranged to record the temperature at each of the nine selected depths in turn, repeating the cycle every 18 min. In this way, practically continuous traces were obtained giving the variations of temperature at each depth. Results obtained during the summer of 1951 covering the period 14 August to 18 September, already discussed by Mortimer (1953), were kindly made available to the present authors by Dr Mortimer for a further analysis. This has been carried out and is now described.

Temperatures in the water column at station *S*, at 12.00 h, 14 August 1951, at the nine fixed depths (0, 4, 7, 10, 14, 19, 24, 31 and 39 m), were read off the traces. By graphical interpolation, a continuous profile giving the variation of temperature with depth was constructed. This procedure was repeated at intervals of $1\frac{1}{2}$ h throughout the observational period 14 August to 18 September 1951. The temperature profiles obtained in this way constitute a series of snapshots of the changing distribution of temperature in the water column at *S*, and, as temperature is a conservative property of the water, the movements of temperature may be used to indicate the movements of water masses. Two of the profiles,

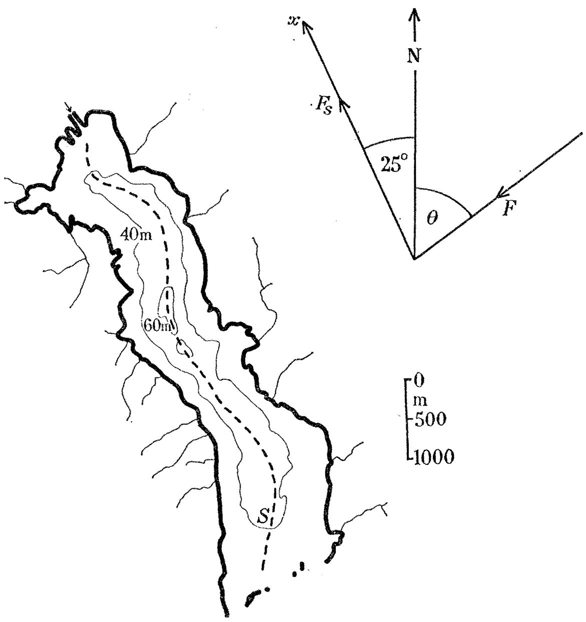

FIGURE 5. Chart of Windermere northern basin showing the 40 and 60 m depth contours (———) and the medial line through the deepest points (– – – –). The resultant wind stress, F, and its component F_S along Ox (parallel to the length of the lake) are indicated.

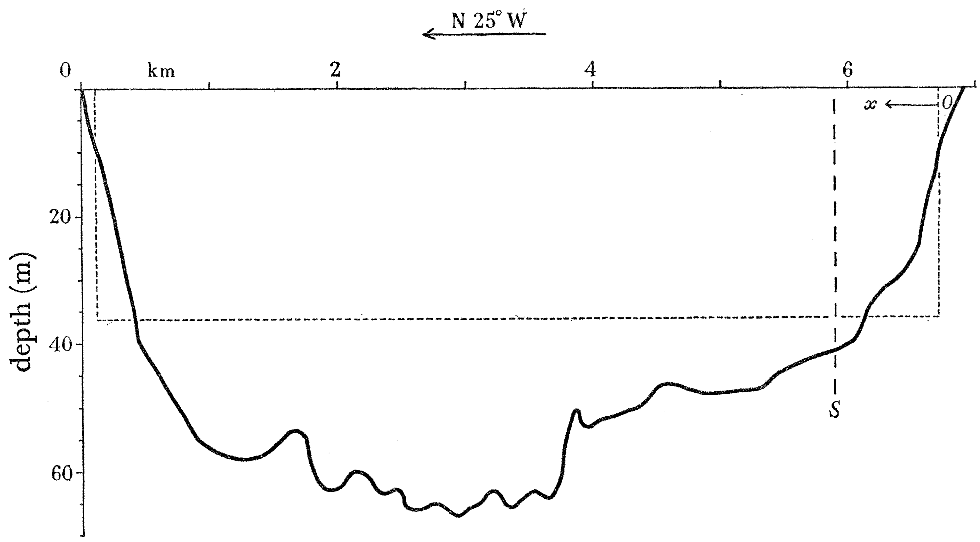

FIGURE 6. Longitudinal section of Windermere northern basin (——); theoretical basin (-----).

shown in figure 7, illustrate the thermal stratification of the lake: a warm surface layer was separated from a colder bottom layer by intermediate water characterized by a large vertical temperature gradient, or *thermocline*. The depths corresponding to 17, 14, 11 and 8 °C (temperatures in the region of the thermocline) were read off each profile and plotted against time. In this way curves were obtained giving the variation in depth of isotherms in the thermocline region at station S. Figure 9 shows the curves for the period 14 to 20 August, and figure 11 the curves for the period 13 to 17 September. These figures are discussed in § 11.

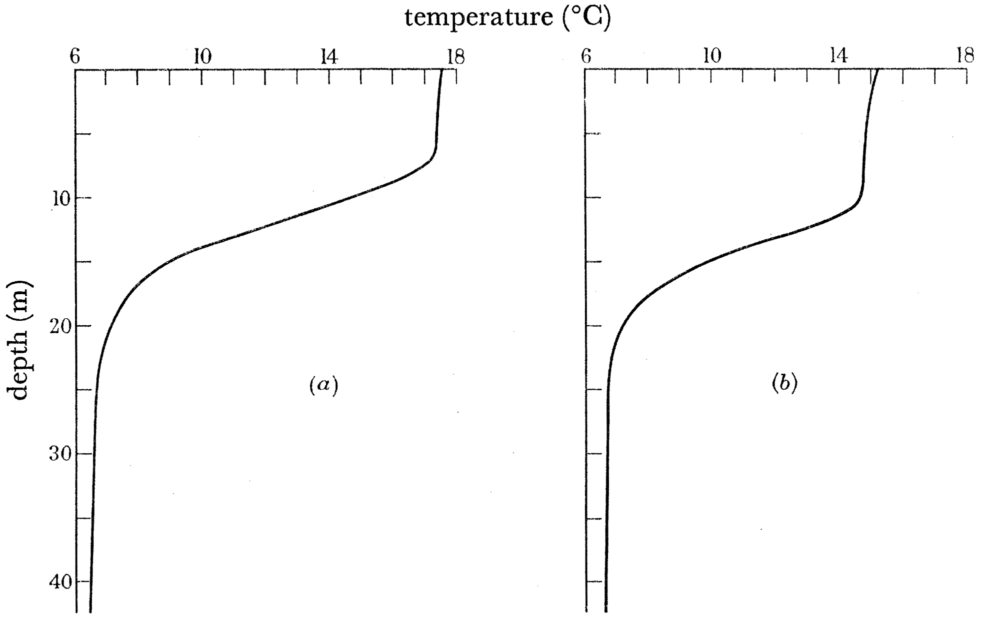

FIGURE 7. Vertical distribution of temperature at station S, Windermere northern basin: (a) 03.00 h, 17 August 1951; (b) 01.30 h, 15 September 1951.

10. Analysis of wind records

For the estimation of wind stress on Windermere northern basin during the period covered by the temperature observations, wind data was available from Dines anemograph records taken at Sellafield and The Ferry House. Sellafield lies on the coast 23 miles west of the lake, while The Ferry House is situated at the southern end of the basin on its western shore. Because of their locations, neither the anemometer at Sellafield nor that at The Ferry House can be relied upon to give an accurate representation of wind conditions over the lake at all times. The Ferry House instrument is sheltered by a hill in the sector southwest to northwest and as a result does not record the full force of the winds from this quarter. This is a serious inconvenience, for many of the winds which affect the lake have a pronounced westerly component. On the other hand, the Sellafield anemometer in its exposed coastal position cannot account for local peculiarities in the wind field over the lake induced by the surrounding hills and side valleys. It seems likely, however, that the records from the latter instrument are the more reliable for the purpose of estimating a mean wind-stress value for the lake at any particular time. Accordingly, stress values derived from the Sellafield records, rather than those derived from the Ferry House records, are used for the main discussion of the present paper.

The anemometer records gave hourly mean values of wind speed and wind veer from the north (angle θ shown in figure 5). The wind stress, F, associated with each wind speed was calculated from the formula:

$$F = c\rho_a U^2,$$

where ρ_a denotes the density of the air, U the wind speed corrected to a height of 10 m, and c the drag coefficient. We took $\rho_a = 0{\cdot}00125\,\mathrm{g/cm^3}$, $c\times 10^3 = 0{\cdot}565$ for $U \leqslant 5$, and $c \times 10^3 = -0{\cdot}12 + 0{\cdot}137U$ for $5 < U < 20$, U here being measured in metres/second. The latter formula for c was communicated to the authors privately by Professor P. A. Sheppard, F.R.S., who inferred the result from wind measurements by application of the well known logarithmic 'law of the wall'.

From each pair of values (F, θ), known at hourly intervals, F_S was calculated from the equation

$$F_S = -F \cos(\theta + 25°),$$

where θ is in degrees. As shown in figure 5, F_S is the wind-stress component over the lake in the direction N 25° W, i.e. the component directed parallel to the length. Values of F_S obtained in this way from the Sellafield records are plotted against time, concurrently with the curves giving the fluctuations in level of isotherms at station S, in figures 9 and 11. Figure 9 covers the time interval 14 to 20 August, and figure 11 the interval 13 to 17 September. Figures 8 and 10 show graphically the steps in the calculation of F_S for each period, respectively: F, θ, and the cosine factor $-\cos(\theta + 25°)$, are plotted against time.

11. Discussion

The temperature records discussed in § 9, covering the period 14 August to 18 September 1951, obtained from the nine thermographs spaced throughout the water column at station S have been presented by Mortimer (1953) in his figure 11. Spells of strong wind occurred from 18 to 19 August, 25 to 26 August, on 29 August, and from 13 to 14 September. During each spell, internal waves, indicated by oscillations of the measured water temperatures, appeared on the thermocline boundary; the waves continued in the form of damped oscillations after the wind had dropped.

As described in § 9, we have analysed the temperature records to determine the fluctuations in depth of isotherms in the thermocline region of the lake. In the discussion which follows, these fluctuations are examined in relation to the variations in the wind-stress component F_S, for the time intervals: 14 to 20 August, 13 to 17 September. The intervals have been selected from the entire observational period (14 August to 18 September) in order to include examples of the internal wave activity mentioned above.

(a) *Observational results*: 14 *to* 20 *August* 1951

During this interval of time there was a relatively calm spell of weather from 14 to 17 August, which was followed by a period of strong wind from 18 to 19 August. The sequence of changes in wind stress (F) and wind direction (θ) is indicated in figure 8. During the calm spell, the wind stress never exceeded 0·6 dyn/cm^2; later, it reached a maximum of 1·7 dyn/cm^2 on 18 August. On 14 and 15 August the wind was in the northwest; its strength decreased progressively, and by the end of 15 August there was an almost perfect calm. From 16 to 18 August the wind was in the south, apart from a temporary swing to the northwest for a few hours on 17 August; on 19 and 20 August it had veered to $\theta \doteqdot 245°$ and was blowing at right angles to the length of the lake.

Figure 9 shows that, in the vertical motion of the thermocline at station S, an oscillation of period 12 h, centred on the 14 °C isotherm, was predominant. This oscillation may be associated with the fundamental internal seiche in a two-layered model of the lake, the interface between the layers of the model corresponding to the 14 °C isotherm in the lake. Shortperiod oscillations of relatively small amplitude are observable, superimposed upon the main 12 h oscillation. Also, a wave with a period exceeding 12 h may be detected in the vertical motion of the 8° isotherm.

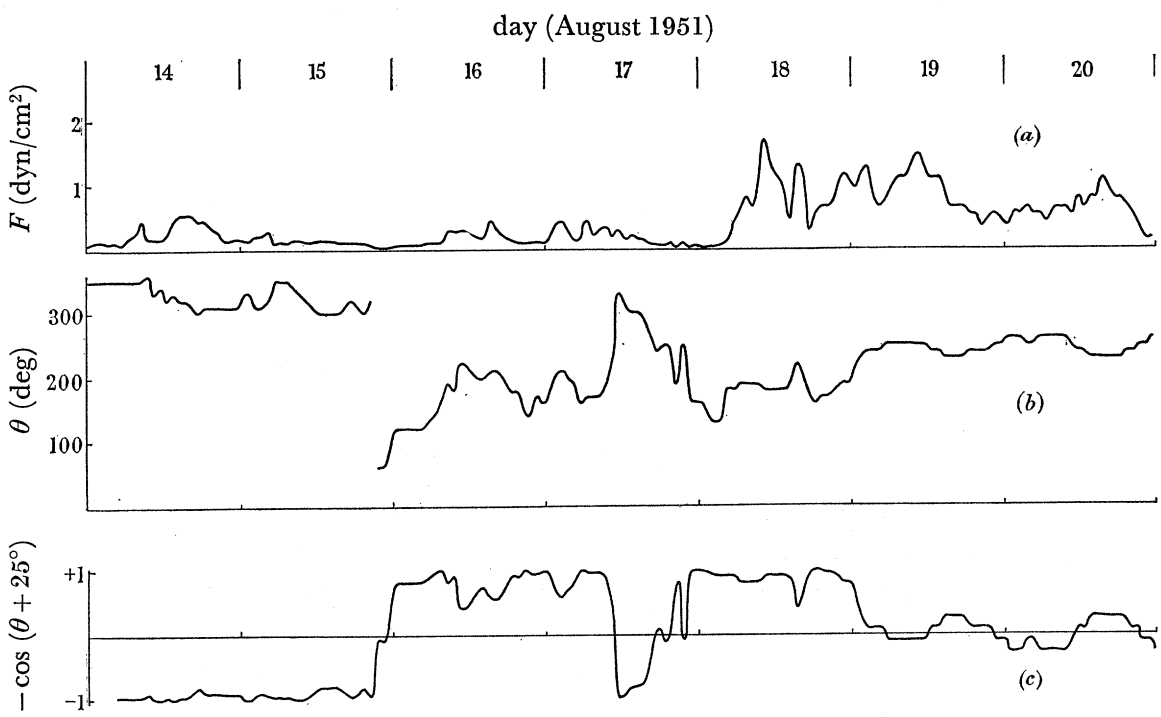

FIGURE 8. (*a*) Resultant wind stress F dyn/cm^2; (*b*) wind direction from the north, θ degrees; and (*c*) the factor $-\cos(\theta+25°)$, for the period 14 August to 20 August 1951.

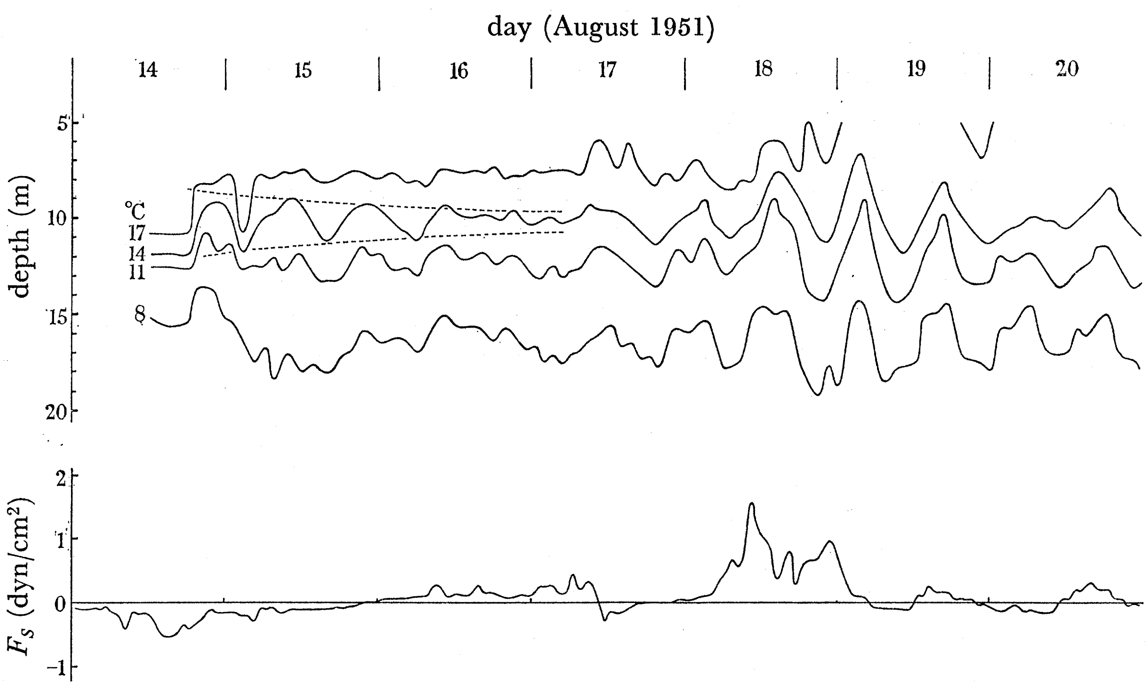

FIGURE 9. Depths of isotherms in the thermocline region at station S of Windermere northern basin, and wind-stress component F_S, during the period 14 August to 20 August 1951.

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From 14 to 17 August, the fluctuations in level of the 14 °C isotherm took the form of a set of damped harmonic oscillations—initiated at the beginning of the period. The amplitude of the oscillations was reduced by about $\frac{1}{3}$ in $2\frac{1}{2}$ days, as indicated by the pecked-line envelope in figure 9: this observation is used in part III to arrive at a value for the frictional coefficient k (equation (33)).

Referring again to figure 9, it is apparent that, in the proximity of 11.00 h on 17 August, there was a reversal in direction of the wind-stress component F_S (F_S was originally positive, directed towards N 25° W, and then became negative, directed towards S 25° E). Subsequent to this reversal, the level of the thermocline at station S took a downward plunge. This may be explained as follows: the northerly wind impulse, applied to the surface of the lake, forced relatively warm water above the thermocline towards the southern end of the basin; the warm surface water, thus transported horizontally, displaced colder bottom water and produced a fall in thermocline level at station S, as observed. The fall in level was clearly the beginning of another set of thermocline oscillations of the type already described. However, before the oscillatory pattern could develop fully, the strong winds on 18 August had a decisive effect on the motion of the isotherms. These winds set up an oscillation on the thermocline boundary in which, as shown in figure 9, the vertical displacement at station S reached as high as 5 m from wave trough to wave crest. Quite apart from the strength of the wind, it would appear that the *timing* of the wind pulses, directed along the length of the lake, contributed significantly to the magnitude of this oscillation: the two main peaks in F_S on 18 August were separated by an interval of 12 h; the 12 h oscillation, excited by the wind pulse associated with the first peak, was thus reinforced by the wind pulse associated with the second peak.

Finally, it is of interest to note the rise in the mean level of the 14° C isotherm during the period of strong wind on 18 and 19 August. This rise, recorded at the southern end of the basin at station S, may be attributed to the 'steady' tilt of the thermocline boundary along the length of the lake, maintained by the wind-stress component F_S directed towards the northern end (e.g. see equation (72)).

(*b*) *Observational and theoretical results*: 13 *to* 17 *September* 1951

The resultant wind stress (F) and the wind direction (θ), during this interval, are plotted against time in figure 10. The stress component F_S derived from these graphs, is given by the continuous-line curve in figure 11. Clearly there were strong winds over the lake during 13 and 14 September, the wind stress reaching a maximum of almost 4 dyn/cm² and exceeding 2 dyn/cm² for most of the time. On 15, 16, and 17 September the winds had moderated and the wind-stress level had dropped to 0·75 dyn/cm² (a mean value).

The changes in wind direction caused significant changes in the sign (i.e. the direction) of F_S. On 13 September the centre of a small but intense secondary depression, moving north-eastwards across the British Isles, passed over the lake. As a result, the wind on that day veered from $\theta = 190°$ at 14.00 h to $\theta = 320°$, 4 h later. This quite rapid change in wind direction, coupled with strong winds before and after the passage of the depression, produced the sharp drop in F_S from $+2{\cdot}2$ dyn/cm² to $-2{\cdot}7$ dyn/cm² apparent in figure 11. Subsequently from 18.00 h on 13 September to 03.00 h on 15 September, the wind backed slowly from $\theta = 320°$ to $\theta = 215°$, and F_S passed from negative to positive values. Under the

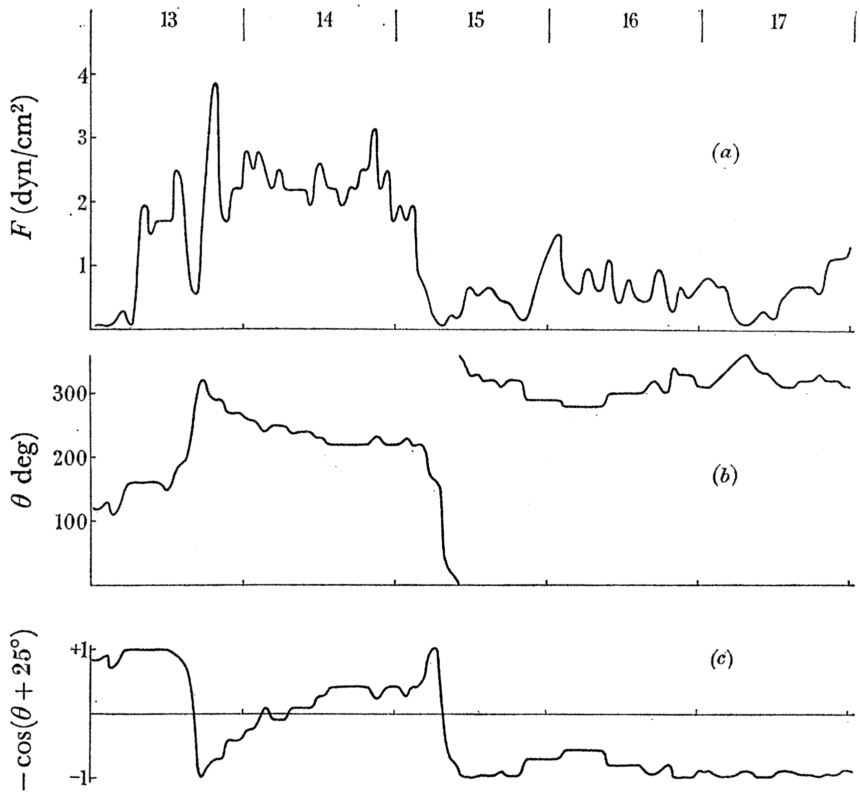

FIGURE 10. (*a*) Resultant wind stress F dyn/cm², (*b*) wind direction from the north, θ degrees, and (*c*) the factor $-\cos(\theta+25^\circ)$, for the period 13 September to 17 September 1951.

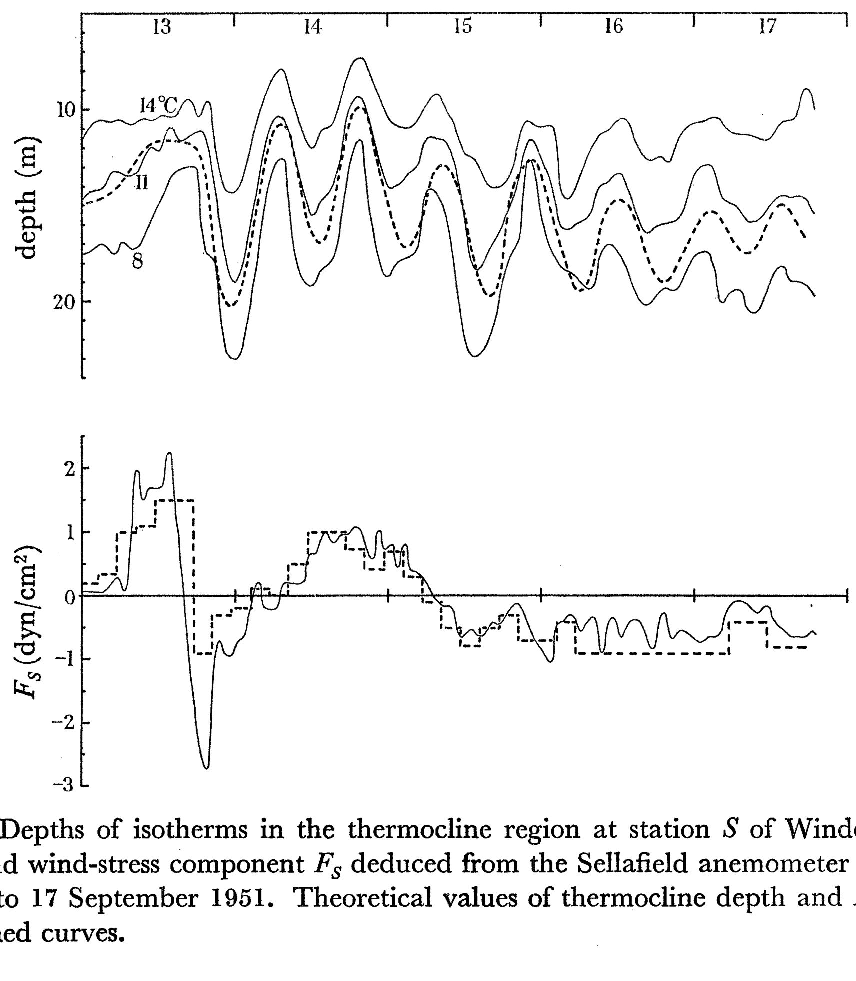

FIGURE 11. Depths of isotherms in the thermocline region at station S of Windermere northern basin and wind-stress component F_S deduced from the Sellafield anemometer records, 13 September to 17 September 1951. Theoretical values of thermocline depth and F_S are shown by the dashed curves.

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influence of a further secondary depression moving eastwards over the British Isles, the wind strength then decreased quickly to almost zero at 07.00 h on 15 September. After this time, the wind rose up in the northwest and F_S became negative.

During the interval, internal waves of large amplitude appeared on the thermocline boundary, as shown by the oscillatory motion of the isotherms at station S (figure 11). One may conclude that these waves were due to the changes in wind stress which took place over the lake *during the interval*, since a week of calm weather preceded the onset of the strong winds on 13 September. Under such conditions, it is appropriate, and of considerable interest, to try to relate the oscillatory motion at S to the changes in the wind-stress component F_S during the interval, using the theory developed in part I. To this end, the theory is applied in part III to determine the fluctuations in the level of the thermocline at S, in response to a time-variation of F_S (taking the form of a series of rectangular wind-stress pulses each of duration 3 h) which approximates to that deduced from the wind records. The fluctuations in level thus obtained, and the F_S variation from which they were derived, are shown by the dashed curves in figure 11. It is apparent from this figure that the thermocline level from theory approximates satisfactorily to a level lying between the 11° and 8° isotherms in the lake.

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PART III. APPLICATION OF THE THEORY TO WINDERMERE

By N. S. Heaps

12. Theory of §5 (numerical example)

The northern basin of Windermere (figure 5) is represented theoretically by a rectangular basin of length 6·6 km and depth 36 m, with the origin at the southern end and the horizontal axis Ox pointing in a direction N 25° W along the length (figure 6). Numerical values, appropriate to the interval 13 to 17 September 1951, are taken as follows:

$$h_1 = 15\,\text{m}, \qquad \rho_1 = 1-8{\cdot}45\times10^{-4}\,\text{g/cm}^3,$$
$$h_2 = 21\,\text{m}, \qquad \rho_2 = 1-0{\cdot}50\times10^{-4}\,\text{g/cm}^3,$$
$$l = 6{\cdot}6\,\text{km}, \qquad g = 980\,\text{cm/s}^2.$$

The values of depth and density have been deduced from the temperature measurements at station S during the interval (figure 11), described in part II; the surface of discontinuity in the two-layered model corresponds approximately to the layer of water in the thermocline region of the real lake between the 11 and 8 °C isotherms. Temperature measurements from 14 to 17 August 1951 show that in a period of calm weather following a wind pulse the amplitude of the vertical thermocline oscillation is reduced by a factor of about $\frac{1}{3}$ in $2\frac{1}{2}$ days (figure 9). To obtain an exponential decay-rate of this order in the formula for ζ_2, given by (86), we take

$$k = 0{\cdot}044\,\text{h}^{-1}.$$

In the absence of data on the wind-stress distribution, $A(x)$, we assume that this is a constant, A_s. If we take

$$\left.\begin{aligned} A(x) &= A_s \quad (0<x<l), \\ &= 0 \quad (x=0, x=l), \end{aligned}\right\}$$

it follows that the Fourier coefficients A_n determined from (61) are

$$\begin{aligned} A_n &= 4A_s/n\pi \quad (n \text{ odd}), \\ &= 0 \qquad\quad (n \text{ even}). \end{aligned}$$

Calculation with the above numerical values, using (80) and (82), gives

$$\epsilon_n = 1{\cdot}817\times10^{-6}/n^2, \quad \delta_{1,n} = 0{\cdot}636\times10^{-6}/n^2, \quad \delta_{2,n} = 1{\cdot}679\times10^{-3}/n^2,$$

and seiche periods

$$2l/n\sigma_{1,n} = 0{\cdot}1952/n, \quad 2l/n\sigma_{2,n} = 14{\cdot}04/n \quad (\text{in hours}).$$

Then it follows from (85) and (86) that in the response of Windermere northern basin to a uniform wind stress F_S of A_s dyn/cm² suddenly created at time $t = 0$ and subsequently maintained,

$$\left.\begin{aligned}
\zeta_1 &= A_s \sum_{r=1}^{\infty} [-0{\cdot}002+0{\cdot}001 \exp(-0{\cdot}02567t)\cos(32{\cdot}19\,nt) \\
&\qquad +0{\cdot}001\exp(-0{\cdot}01833t)\cos(0{\cdot}4475nt)]\,\frac{\cos(0{\cdot}476nx)}{n^2}, \\
\zeta_2 &= A_s \sum_{r=1}^{\infty} [2{\cdot}289-\exp(-0{\cdot}01833t)\{2{\cdot}289\cos(0{\cdot}4475\,nt) \\
&\qquad +(0{\cdot}094/n)\sin(0{\cdot}4475nt)\}]\,\frac{\cos(0{\cdot}476nx)}{n^2}, \\
u_{1m} &= A_s \sum_{r=1}^{\infty} [0{\cdot}040\exp(-0{\cdot}02567t)\sin(32{\cdot}19nt) \\
&\qquad +3{\cdot}987\exp(-0{\cdot}01833t)\sin(0{\cdot}4475nt)]\,\frac{\sin(0{\cdot}476nx)}{n^2}, \\
u_{2m} &= A_s \sum_{r=1}^{\infty} [0{\cdot}040\exp(-0{\cdot}02567t)\sin(32{\cdot}19nt) \\
&\qquad -2{\cdot}845\exp(-0{\cdot}01833t)\sin(0{\cdot}4475nt)]\,\frac{\sin(0{\cdot}476nx)}{n^2},
\end{aligned}\right\} \quad (125)$$

where x is measured in kilometres, t in hours, ζ_1, ζ_2 in metres, and u_{1m}, u_{2m} in centimetres/second; $n = 2r-1$. By definition, F_S in the component of the resultant wind stress along Ox, i.e. in the direction N 25° W parallel to the length of the basin. Taking $A_s = 1$ dyn/cm² in (125), ζ_2 (at $x/l = 0{\cdot}126$, station S) and u_{1m}, u_{2m} (at $x/l = 0{\cdot}5$) are plotted against time in figure 12. Inspection of this figure shows that the greatest value of u_{2m} is 3 cm/s, which, using (33), leads to a value of 0·15 dyn/cm² for the bottom friction. Hence, at the mid-way station of the lake, $x/l = 0{\cdot}5$, the ratio of bottom stress to wind stress does not exceed 0·15. Combining, as in (124), forms of ζ_2 obtained by taking various values of A_s and different time-origins in (125), ζ_2 is obtained which, at station S, approximates closely to the fluctuations in the level of the thermocline observed there from 13 to 17 September 1951 (figure 11). The wind stress F_S associated with this variation of ζ_2 approximates to F_S estimated from anemometer records. (The high negative peak in F_S on 13 September is only partially reproduced in the theoretical representation: as an explanation of this result, it is suggested that the full height of the peak—deduced from the Sellafield wind records—may not have been realized over the lake due to the effect of the surrounding hills.) The depth-mean velocities u_{1m}, u_{2m} at the mid-position of the lake ($x/l = 0{\cdot}5$), obtained from (125) like ζ_2, are plotted against time in figure 13.

13. Theory of §6 (numerical example)

The theory given in §6 is now applied to Windermere northern basin. For simplicity, only ζ_2 is determined and consideration is restricted to the case when $F_S = H(t)\,A_1 \sin \pi x/l$ ($n = 1$). There are then two (uninodal) seiches in the lake: one ordinary and the other internal. The numerical values of h_1, h_2, ρ_1, ρ_2, l, and g already introduced (corresponding to natural conditions from 13 to 17 September, 1951) are again employed. From (102) and (104),

$$b = 1{\cdot}7143, \quad d = 5{\cdot}6786\times 10^{-4}.$$

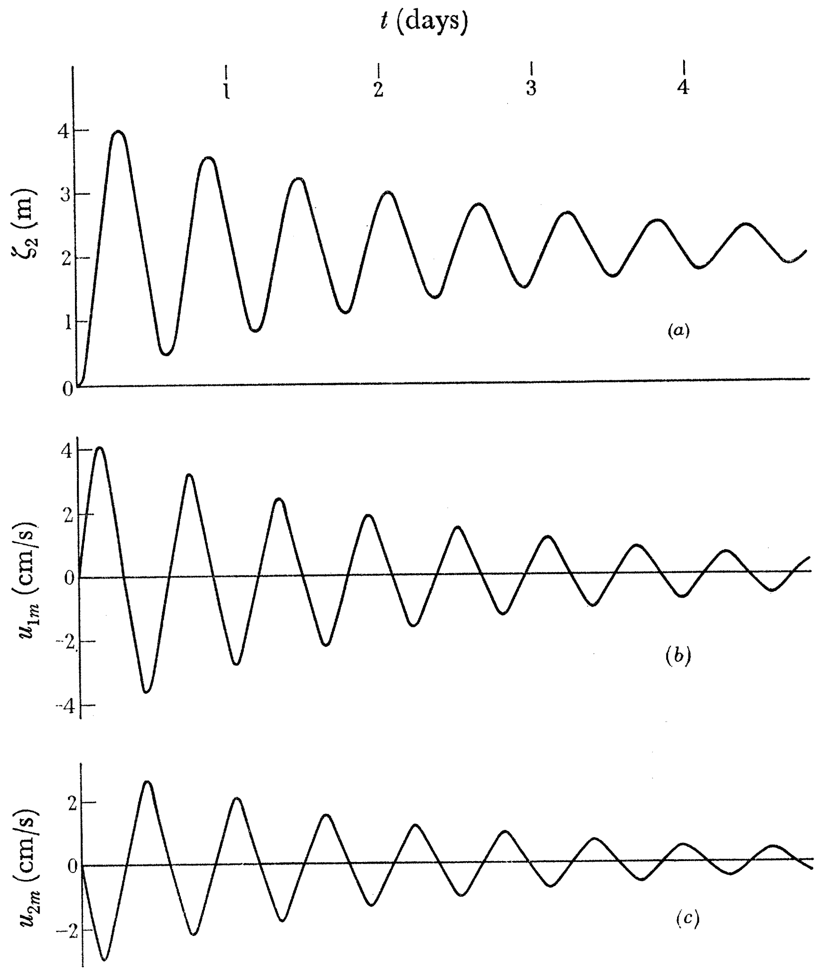

FIGURE 12. Elevation ζ_2 and depth-mean currents u_{1m}, u_{2m} corresponding to a uniform wind-stress component F_S of 1 dyn/cm^2, suddenly created at $t = 0$ over Windermere northern basin (theoretical results): (*a*) ζ_2 (m) at $x/l = 0{\cdot}126$ (station *S*), (*b*) u_{1m} (cm/s) at $x/l = 0{\cdot}5$, (*c*) u_{2m} (cm/s) at $x/l = 0{\cdot}5$.

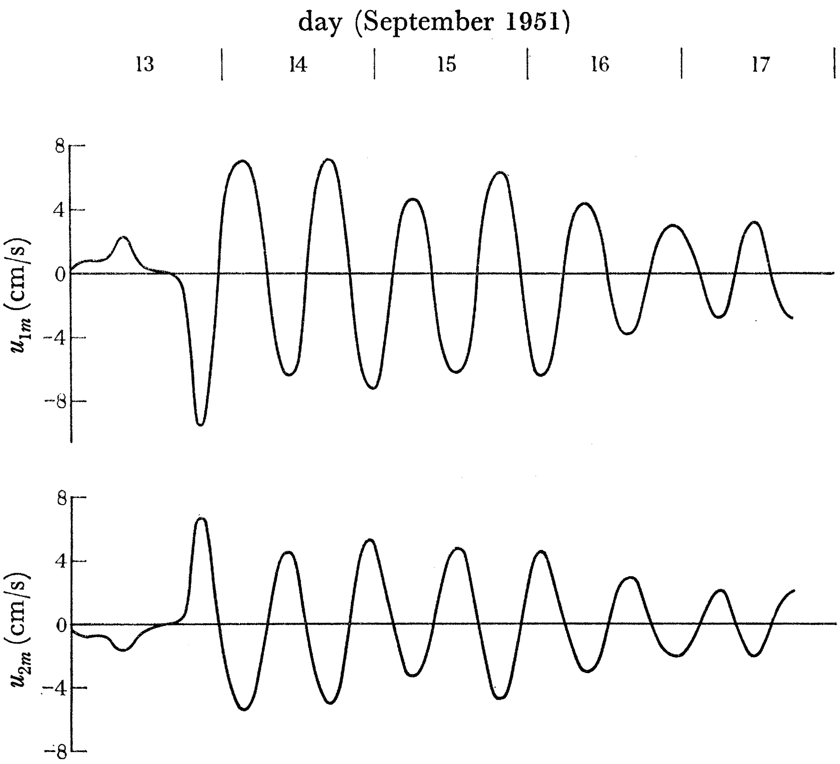

FIGURE 13. Depth-mean currents u_{1m}, u_{2m} (cm/s) at the mid-position ($x/l = 0{\cdot}5$) of Windermere northern basin, for the period 13 September to 17 September 1951 (theoretical results).

Using (105), a is plotted against ν for $0 < \nu < \frac{1}{2}\pi$; reading the minimum value of a off the graph gives

$$a_0 = 0{\cdot}919 \times 10^{-3}.$$

It follows from (120) that friction in the bottom layer of the lake is great enough to prevent the development of the internal seiche if

$$N_2 \geqslant 913\ \text{cm}^2/\text{s}.$$

The appropriate value of N_2 for Windermere under the conditions being considered is therefore less than this value. In our calculations equation (99) is solved to find κ_1, τ_1, κ_2, τ_2 when N_2 (cm²/s) = 1, 2·5, 10, 26, 50, 100 (100) 900. For each N_2 the seiche periods T_1, T_2 are determined by using (117), and the corresponding damping coefficients γ_1, γ_2 using (114). The results are shown in table 1. [The roots $\kappa_1 + \mathrm{i}\tau_1$, $\kappa_2 + \mathrm{i}\tau_2$ are determined for a particular N_2 by using Newton's method of successive approximation. First approximations to the roots for $N_2 = 1$ may be chosen as follows:

$$\kappa_1 + \mathrm{i}\tau_1 \doteqdot (1+\mathrm{i})\,(h/4ah_2)^{\frac{1}{4}} = 140{\cdot}4\,(1+\mathrm{i}),$$

$$\kappa_2 + \mathrm{i}\tau_2 \doteqdot (1+\mathrm{i})\,(ch_2/4h)^{\frac{1}{4}} = 16{\cdot}55\,(1+\mathrm{i}),$$

these being values of ν corresponding to the roots of $\Phi(\lambda) = 0$ (see (51) and (69); λ is related to ν by (106)). The values of $\kappa_1 + \mathrm{i}\tau_1$, $\kappa_2 + \mathrm{i}\tau_2$ found for $N_2 = 1$ may be used as first approximations in their determination for $N_2 = 2{\cdot}5$, and so on for all the values of N_2 taken in ascending order of magnitude.] Table 1 leads us to take $N_2 = 26$, for this gives a damping coefficient of 0·018h⁻¹ in the internal seiche mode, as evidenced by (125). In this case, from (101) and (103),

$$a = 0{\cdot}74544 \times 10^{-6}, \quad c = 0{\cdot}76177 \times 10^{3},$$

Table 1. Values of T_1, T_2, γ_1, γ_2, corresponding to various values of N_2

N_2 (cm²/s)	T_1 (h)	T_2 (h)	γ_1 (h⁻¹)	γ_2 (h⁻¹)
1	0·1954	14·13	0·0337	0·002953
2·5	0·1956	14·18	0·0530	0·004801
10	0·1959	14·32	0·1064	0·01036
26	0·1963	14·48	0·1723	0·01822
50	0·1967	14·60	0·2400	0·02800
100	0·1973	14·71	0·3414	0·04957
200	0·1981	15·00	0·4870	0·09530
300	0·1988	15·50	0·6006	0·1418
400	0·1994	16·29	0·6975	0·1886
500	0·2000	17·50	0·7838	0·2355
600	0·2003	19·42	0·8625	0·2824
700	0·2007	22·80	0·9356	0·3294
800	0·2011	30·36	1·004	0·3764
900	0·2015	86·91	1·069	0·4234

from (99),

$$\kappa_1 = 27{\cdot}536, \quad \tau_1 = 27{\cdot}388,$$

$$\kappa_2 = 3{\cdot}2646, \quad \tau_2 = 3{\cdot}1304,$$

and from (110) and (111),

$$L_2(\kappa_1, \tau_1) = -0{\cdot}000097, \quad M_2(\kappa_1, \tau_1) = -0{\cdot}0000006,$$

$$L_2(\kappa_2, \tau_2) = 0{\cdot}50616, \quad M_2(\kappa_2, \tau_2) = 0{\cdot}01295.$$

The functions $G_2(\kappa_1, \tau_1)$, $G_2(\kappa_2, \tau_2)$ given by (113) are thus determined. Solving (99) for the real root ν_1 we get

$$\nu_1 = 4\cdot5387,$$

so that

$$N_2\nu_1^2/h_2^2 = 0\cdot4372\text{h}^{-1}$$

and, from (108),

$$H_2(\nu_1) = -0\cdot0104.$$

With these numerical values it follows from (115) and (116) that

$$\begin{aligned}\zeta_2 = 2\cdot289\,(\tfrac{1}{4}\pi A_1)\,[1 &- \{1\cdot0123\cos(0\cdot4338t) + 0\cdot0259\sin(0\cdot4338t)\}\exp(-0\cdot01822t)\\ &+\{0\cdot00019\cos(32\cdot01t)+0\cdot000001\sin(32\cdot01t)\}\exp(-0\cdot1723t)\\ &+0\cdot0104\exp(-0\cdot4372t)+\ldots]\cos(0\cdot476x),\end{aligned} \tag{126}$$

where x is in kilometres, t in hours, and ζ_2 in metres. The uninodal part of ζ_2 given by (125), and ζ_2 given by (126) with $A_1 = 4A_s/\pi$, both approximate to

$$2\cdot289A_s[1-\cos(0\cdot44t)\exp(-0\cdot018t)]\cos(0\cdot476x),$$

i.e. taking $F_S = (4A_s/\pi)\,H(t)\sin(\pi x/l)$, the two theories considered, when applied to Windermere, have yielded approximately the same expression for ζ_2. This gives support to the choice: $N_2 = 26$.

14. Conclusions

A theory is given for the motion in a long narrow lake subject to wind stress during the period of summer stratification. (Conditions in the lake then approximate to a two-layered system: a warm surface layer, the epilimnion, lies over a colder bottom layer, the hypolimnion; between them is a large vertical temperature gradient, the thermocline.)

By means of the theory, the fluctuations in level of the thermocline at any position, and the currents in the lake, may be deduced from the variations in wind stress over the water surface estimated from wind-speed observations. It appears that there are scarcely any reliable current measurements against which theoretically derived values may be tested—which emphasizes the need for a current meter sensitive enough to measure velocities of not more than a few centimetres per second. However, fluctuations in thermocline level obtained using the theory may be compared with those deduced from temperature measurements. This has been done for Windermere northern basin, covering a period of five days, and there is satisfactory agreement between theory and observation.

The mathematical analysis indicates that a change in the wind-stress component parallel to the length of the lake sets up longitudinal internal seiches centred on the thermocline and longitudinal surface seiches; the extent to which the seiches of different nodality are excited depends on the distribution of the wind-stress change along the lake. According to the analysis, wind-driven circulations are confined to the surface layer: there is no friction at the internal boundary by which this motion may be transmitted into the bottom layer. For the same reason, the horizontal shearing effect of bottom friction does not reach up across the boundary into the top layer.

The seiches are damped out by friction—in the mathematical model this is applied externally at the bottom of the lake. Taking the bottom friction as proportional to the horizontal flow in the bottom layer leads to a relatively simple solution of the hydrodynamical equations; employing instead, the condition of zero bottom current, leads to a more general solution, but lengthy computations are required for its numerical evaluation.

The theory indicates that, under certain conditions, internal seiches do not develop in response to wind disturbances. This is due to the damping effect of friction in the bottom layer and is likely to occur in the autumn, just before the overturn, when the density difference between the layers decreases and the depth of the surface layer increases.

In the study of the dynamics of a thermally stratified lake under wind stress, more detailed observation is required to make a thorough check on theory. In planning a further series of observations, consideration might be given to the possibility of measuring continuously, over an extended period of time: (i) temperatures and currents, throughout the depth of water, at more than one station; (ii) wind speed and direction at various locations over the surface of the lake. The magnitude of such an undertaking should not be underestimated. The cost of instruments, and the labour involved in their deployment and use, would be considerable. The practical difficulties in attempting to measure small currents would have to be overcome. The large amount of observational data obtained would require careful analysis.

A revision or extension of the present theory might become necessary in the light of more detailed observational results. In particular, the frictional condition at the thermocline boundary might require some modification. For larger lakes, the theory would have to be extended to take into account the effect of the earth's rotation (Mortimer 1963).

Grateful acknowledgement is made to Professor J. Proudman, F.R.S., Dr C. H. Mortimer, F.R.S., and Dr A. T. Doodson, F.R.S. for their comments on the manuscript of this paper.

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